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SOME NEW IDEALS OF SETS ON THE REAL LINE

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0. This paper is inspired by a work of Schmidt [9]. Theorem 2 of [9] shows the existence of new ideals of sets of real numbers. Like the classical ideals of sets of measure 0 and of sets of the first category these ideals are not trivial, i.e., they do not contain all sets of reals, they are countably additive, they are invariant under linear non-singular transformations of the real line R and they contain sets which are large from the point of view of the classical ideals. In fact, R is the union of two Borel sets one of measure 0 and of the first category and the other belongs to the ideals of Schmidt. (Recall the existence of a similar partition of R into a set of measure 0 and a set of the first category.)

S. M. Ulam told me that there are many non-trivial countably additive ideals of sets in R^n which are invariant under all isometries and are not included in the classical ideals. E.g., all sets in R^2 which are of the first category on almost all lines, given a "Lebesgue" measure in the space of lines, constitute such an ideal.

In this paper I list several properties and several open problems concerning the ideals of Schmidt. Section 3 concerns also other ideals of sets in topological spaces.

1. For technical simplicity I will modify somewhat the concepts of [9]. I will not use the real line but the compact topological group C which is the direct product of ω copies of the cyclic two element group $\langle\{0, 1\}, +\rangle$ with discrete topology. Thus C is homeomorphic and can be identified to the set of Cantor $\{\sum_{i=0}^{\infty} 2x_i/3^{i+1}\colon x_i = 0, 1\}$, and the Haar measure in C is induced by the Lebesgue measure over the unit interval I and the Cantor mapping of C onto I.

Given a set $S \subseteq C$ and a set K of natural numbers we define a positional game $\Gamma(S, K)$ with perfect information between two players I and II (1). The players choose the consecutive terms of a sequence

(1) See [5], § 2, for a more general treatment of such games.

$(x_0, x_1, x_2, \ldots) \in C$, the choice x_i is done by Player I if $i \notin K$ and by Player II if $i \in K$. The player choosing x_i knows S, K and $x_0, \ldots, x_{i-1}$. Player I wins if $(x_0, x_1, \ldots) \in S$ and Player II wins in the other case.

Let $V_{II}(K)$ denote the class of sets $S \subseteq C$ for which Player II has a winning strategy in the game $\Gamma(S, K)$.

Let $M = \langle K_{s_1,\ldots,s_n} : s_i = 0, 1;\ n = 1, 2, \ldots \rangle$ be a system of sets of natural numbers such that $K_{s_1,\ldots,s_n,s_{n+1}} \subseteq K_{s_1,\ldots,s_n}$ and

$$K_{s_1,\ldots,s_{n-1},0} \cap K_{s_1,\ldots,s_{n-1},1} = 0 \quad \text{for } n = 1, 2, \ldots$$

We put

$$I_M = \bigcap V_{II}(K_{s_1,\ldots,s_n}),$$

where $K_{s_1,\ldots,s_n}$ runs over all the sets of the system M.

2. The following proposition follows directly from the definition of I_M:

PROPOSITION 1. *I_M is translation invariant, i.e., for every $S \in I_M$ and every $x \in C$ we have $S + x \in I_M$.*

PROPOSITION 2. (i) *I_M is hereditary, i.e., for every $S \in I_M$ and every $T \subset S$ we have $T \in I_M$.*

(ii) *No open non-empty set belongs to I_M.*

Proof. (i) is obvious from the definition of I_M.

(ii) Let V be an open set in C and let $(x_0^0, x_1^0, \ldots, x_n^0)$ be a sequence such that $(x_0^0, x_1^0, \ldots, x_n^0, x_{n+1}, x_{n+2}, \ldots) \in V$ for every $x_{n+1} = 0, 1$. Since the sets $K_{s_1,\ldots,s_n}$ are disjoint and there are 2^n such sets then one of them, say $K_{s_1^0,\ldots,s_n^0}$, is disjoint with the set $\{0, 1, \ldots, n\}$. Hence the game $\Gamma(V, K_{s_1^0,\ldots,s_n^0})$ is a win for I and (ii) follows.

THEOREM 3. *I_M is a countably additive ideal of sets* [2].

Proof. Let $S_i \in I_M$ for $i = 1, 2, \ldots$ Then Player II has a winning strategy $\sigma_{i,s_1,\ldots,s_m}$ for each of the games $\Gamma(S_i, K_{s_1,\ldots,s_m})$. Then he has a winning strategy in the game $\Gamma_0 = \Gamma(\bigcup_{i=1}^{\infty} S_i, K_{s_1^0,\ldots,s_n^0})$. Indeed, whenever he is making his j-th choice, where $j \in K_{s_1^0,\ldots,s_n^0,s_1,\ldots,s_i}$ and $s_1 = s_2 = \ldots = s_{i-1} = 0$ and $s_i = 1$ let him play according to the strategy $\sigma_{i,s_1^0,\ldots,s_n^0,s_1,\ldots,s_i}$. Of course this is a winning strategy for Γ_0. Since the choice of $s_1^0, \ldots, s_n^0$ in Γ_0 was arbitrary we see that $\bigcup_{i=1}^{\infty} S_i \in I_M$. Then by Proposition 2 (i) we get the conclusion.

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[2] Compare Theorem 2 in [9] and proposition (i) in [6], p. 209.

THEOREM 4. *If all the sets $K_{s_1,\ldots,s_n}$ are non-empty, then $\boldsymbol{I}_M$ contains all countable subsets of C* [3].

Proof. By Theorem 3 it is enough to prove that $\boldsymbol{I}_M$ contains all singletons. Indeed, if II has at least one move he can avoid any given point.

Let S_M denote the set of all $x = (x_0, x_1, \ldots) \in C$ such that for every $K_{s_1,\ldots,s_n}$ of M there exists some $i \in K_{s_1,\ldots,s_n}$ with $x_i = 0$.

If all the sets $K_{s_1,\ldots,s_n}$ are non-empty (and hence infinite), then S_M is of measure 1 and is an intersection of a countable collection of open sets, all dense in C. Hence $C \setminus S_M$ is of measure 0 and of the first category. Still the following proposition holds:

PROPOSITION 5. $S_M \in \boldsymbol{I}_M$ [4].

Proof. Playing a game $\Gamma(S_M, K_{s_1,\ldots,s_n})$ Player II will win if he always chooses 1.

3. Let $\boldsymbol{I}$ be an ideal of subsets of a topological space X. For every $S \subseteq X$ we denote by $S^{(\boldsymbol{I})}$ the set of all $x \in X$ such that for every neighborhood V of x we have $V \cap S \notin \boldsymbol{I}$.

The following properties of the operation $S^{(\boldsymbol{I})}$ are given by Kuratowski in [3], § 7, IV, V.

PROPOSITION 6. (i) *$S^{(\boldsymbol{I})}$ is closed and is included in the closure of S;*

(ii) *$S \subseteq T$ implies $S^{(\boldsymbol{I})} \subseteq T^{(\boldsymbol{I})}$;*

(iii) $(S^{(\boldsymbol{I})})^{(\boldsymbol{I})} \subseteq S^{(\boldsymbol{I})}$;

(iv) *if G is open, then $G \cap S^{(\boldsymbol{I})} = (G \cap S)^{(\boldsymbol{I})}$;*

(v) $(\bigcap_{a \in A} S_a)^{(\boldsymbol{I})} \subseteq \bigcap_{a \in A} (S_a)^{(\boldsymbol{I})}$ *and* $\bigcup_{a \in A} (S_a)^{(\boldsymbol{I})} \subseteq (\bigcup_{a \in A} S_a)^{(\boldsymbol{I})}$, *where A is any set of indices;*

(vi) $(S \cup T)^{(\boldsymbol{I})} = S^{(\boldsymbol{I})} \cup T^{(\boldsymbol{I})}$;

(vii) $S^{(\boldsymbol{I})} \setminus T^{(\boldsymbol{I})} \subseteq (S \setminus T)^{(\boldsymbol{I})}$;

(viii) $\big((S \setminus T_1) \cup T_2\big)^{(\boldsymbol{I})} = S^{(\boldsymbol{I})}$ if $T_1, T_2 \in \boldsymbol{I}$.

We have also the following proposition:

PROPOSITION 7 [5]. *If X has a countable base of open sets and $\boldsymbol{I}$ is countably additive, then*

(i) $S \setminus S^{(\boldsymbol{I})} \in \boldsymbol{I}$;

(ii) *$S^{(\boldsymbol{I})} = 0$ iff $S \in \boldsymbol{I}$;*

(iii) $(S^{(\boldsymbol{I})})^{(\boldsymbol{I})} = S^{(\boldsymbol{I})}$;

(iv) $(S \cap S^{(\boldsymbol{I})})^{(\boldsymbol{I})} = S^{(\boldsymbol{I})}$.

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(3) Compare Lemma 14 in [9].

(4) Compare Theorems 3 and 5 in [8].

(5) Compare [3], § 10, V, 7, 8, 9, and § 18, IV, (8).

Proof. (i) Every $x \in S \setminus S^{(I)}$ has a neighborhood V_x such that $V_x \cap S \in \boldsymbol{I}$. There exists a countable sequence of open sets $V_1, V_2, \ldots$ such that for every i there exists an x with $V_i \subseteq V_x$ and $\bigcup V_i = \bigcup V_x$. Hence $S \setminus S^{(I)} \subseteq \bigcup_{i=1}^{\infty} (V_i \cap S) \in \boldsymbol{I}$.

(ii) If $S^{(I)} = 0$, then $S = S \setminus S^{(I)} \in \boldsymbol{I}$ by (i). The converse implication is obvious.

(iii) By 6(vii) and 7(i), (ii) we have $S^{(I)} \setminus (S^{(I)})^{(I)} \subseteq (S \setminus S^{(I)})^{(I)} = 0$, i.e., $S^{(I)} \subseteq (S^{(I)})^{(I)}$. The converse inclusion is 6(iii).

(iv) $S = (S \setminus S^{(I)}) \cup (S \cap S^{(I)})$, hence relation (iv) follows from 6(vi) and 7(i).

Remark. If X is compact, then 7 (ii) holds without assuming the suppositions of 7. In fact, if $S^{(I)} = 0$, then every $x \in X$ has a neighborhood V_x such that $V_x \cap S \in \boldsymbol{I}$. X is a finite union of some sets V_x and S is a union of the corresponding sets $V_x \cap S$. Hence $S \in \boldsymbol{I}$, because $\boldsymbol{I}$ is additive. The converse implication is obvious.

We put $S^{[I]} = S \cup S^{(I)}$.

PROPOSITION 8 (6). *If X has a countable base and $\boldsymbol{I}$ is countably additive and contains all singletons, then*

(i) *$S^{[I]}$ is a topological closure operation, i.e., $(S \cup T)^{[I]} = S^{[I]} \cup T^{[I]}$, $(S^{[I]})^{[I]} = S^{[I]}$, $\{x\}^{[I]} = \{x\}$ and $0^{[I]} = 0$;*

(ii) *the topology $\boldsymbol{T}$ defined by this operation includes the original topology of X;*

(iii) *the Borel sets of the topology $\boldsymbol{T}$ are of the form $(B \setminus S_1) \cup S_2$, where B is Borel in the original topology and $S_1, S_2 \in \boldsymbol{I}$.*

Proof. (i) By 6(vi) and 7(iii).

(ii) By 6 (i) if S is closed in X, then $S^{[I]} = S$, i.e., S is closed in $\boldsymbol{T}$.

(iii) The class of sets of the required form $(B \setminus S_1) \cup S_2$ is closed under complementation and countable union. Hence it remains to show that all closed sets of $\boldsymbol{T}$ are of this form. Indeed, $S^{[I]} = S^{(I)} \cup (S \setminus S^{(I)})$, where $S^{(I)}$ is closed in X by 6 (i) and $S \setminus S^{(I)} \in \boldsymbol{I}$ by 7(i).

4. Now we come back to our space C and ideals $\boldsymbol{I}_M$.

THEOREM 9. *If $S \notin \boldsymbol{I}_M$ and $S \in \boldsymbol{F}_{\sigma\delta} \cup \boldsymbol{G}_{\delta\sigma}$, then there exists a closed set $F \subseteq S$ such that $F \notin \boldsymbol{I}_M$.*

Proof. If $S \notin \boldsymbol{I}_M$, then Player II has no winning strategy in one of the games $\Gamma(S, K_{s_1,\ldots,s_n})$. Then, since $S \in \boldsymbol{F}_{\sigma\delta} \cup \boldsymbol{G}_{\delta\sigma}$, by a theorem of Davis [1] Player I has a winning strategy σ. Let F be the set of all $x \in C$ which may result when I uses σ. Clearly, F satisfies the conclusion.

(6) Compare [6], footnote (8).

PROBLEM 1. It is not known if Theorem 9 and the theorem of M. Davis on which it is based can be extended to all Borel sets ([7]). (**P 646**)

THEOREM 10. *For every set* $S \in \boldsymbol{I}_M$ *there exists a set* $A \in \boldsymbol{G}_\delta$ *such that* $S \subseteq A \in \boldsymbol{I}_M$.

Proof. $S \in \boldsymbol{I}_M$ iff II has a winning strategy $\sigma_{s_1,\ldots,s_n}$ for each game $\Gamma(S, K_{s_1,\ldots,s_n})$. Let $F_{s_1,\ldots,s_n}$ be the set of all plays which may result when II uses $\sigma_{s_1,\ldots,s_n}$. Put $A = C \setminus \bigcup F_{s_1,\ldots,s_n}$ and the conclusion follows.

PROBLEM 2. Does there exist for every set $Z \subseteq C$ a Borel set $B \supseteq Z$ such that for every Borel set $B_1 \subseteq Z$ we have $B \setminus B_1 \in \boldsymbol{I}_M$? (**P 647**)

By a general theorem of Marczewski [4] (see also [3], § 11, VII) and our 8 (iii) an affirmative answer to this problem would imply that the class of sets of the form $(B \setminus S_1) \cup S_2$, where B is Borel and $S_1, S_2 \in \boldsymbol{I}_M$, is invariant under the operation $(\mathscr{A})$. Notice the well known facts that if we substitute $\boldsymbol{I}_M$ by the ideal of sets of measure 0 or the ideal of sets of the first category, then the answer is yes.

Theorems 9 and 10 point out similarities between $\boldsymbol{I}_M$, measure 0 and first category. The next propositions show dissimilarities.

By the definition of M the sets K_0 and K_1 are disjoint and hence the complement of one of them, say of K_r, is infinite. Let σ be any strategy of Player I in a game $\Gamma(S, K_r)$ and F be the set of all $x \in C$ which may result when I uses σ. Hence F is closed and $F + x \notin \boldsymbol{I}_M$ for every $x \in C$.

Let D be the set of all $x \in C$ with $x_i = 0$ for all $i \in K_r$. Hence D is a perfect set and hence has the power $2^{\aleph_0}$. From these definitions immediately follows

PROPOSITION 11. $(F + x_1) \cap (F + x_2) = 0$ *for all* $x_1, x_2 \in \boldsymbol{D}$, $x_1 \neq x_2$.

Thus there exists $2^{\aleph_0}$ disjoint closed sets none of which is in $\boldsymbol{I}_M$.

Let E be the set of all $x \in C$ for which there exists some n such that $x_i = 0$ for all $i > n$.

PROPOSITION 12. $C \setminus (F + E) \notin \boldsymbol{I}_M$.

Proof. Since Player I has infinitely many moves in the game $\Gamma(C \setminus (F+E), K_r)$, he can avoid one by one all the sets $F + x$, where x runs over the countable set E. Hence this game is a win for I and our proposition follows.

Since $F + x \notin \boldsymbol{I}_M$, proposition 12 shows that the 01-law fails to hold true for the ideal $\boldsymbol{I}_M$.

Let F_0 be the set of all $x \in C$ such that $x_i = 0$ for all $i \notin K_r$, i.e., F_0 is the set of all x which may result in a game $\Gamma(S, K_r)$ when Player I uses the strategy to choose always 0. Then we obviously have

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([7]) A recent not yet published result of Anthony Martin is that assuming the existence of 01-measurable cardinals both results are valid for all sets which are analytic or a complement of analytic.

PROPOSITION 13. $F_0+F_0 = F_0-F_0 = F_0 \notin I_M$.

This shows that the theorem of Steinhaus, which says that for every set F which is measurable and of positive measure or has the property of Baire and is not of the first category $F-F$ contains an open set, fails to hold for the ideal I_M (because F_0 is nowhere dense).

PROBLEM 3. Let $S \in I_M$. Does there exist a perfect set $P \subseteq C$ such that $x-y \notin S$ for every $x, y \in P$, $x \neq y$? (**P 648**)

For the ideals of sets of measure 0 or of the first category the answers are affirmative (see [7] and [8]). If I is one of these ideals, then the class of sets of the form $(B \setminus S_1) \cup S_2$ considered in Proposition 8 (ii) coincides with the class of measurable sets or sets having the property of Baire, respectively. We know that there are sets in C which are not measurable and do not have the property of Baire.

PROBLEM 4. Does this class differ from the class of all subsets of C also in the case when $I = I_M$? (**P 649**)

There exists a maximal filter of closed subsets of the real line R which, moreover, does not contain any set of finite measure (any maximal extension of the filter of all closed subsets of R which have complements of finite measure is such a filter) or does not contain any nowhere dense set (this was proved on account of the continuum hypothesis by Fine and Gillman [2]).

PROBLEM 5. Does there exist a maximal filter of closed subsets of C which, moreover, is disjoint with I_M? (**P 650**)

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