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ON EXTENSIONS OF RÉNYI CONDITIONAL PROBABILITY SPACES

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0. Introduction. In papers [2] and [3] (see also [4]), Alfred Rényi presented a new approach to probability theory, based on a concept of conditional probability. The conditional probability is defined there as a function $P\colon \mathscr{A}\times\mathscr{B}\to[0, 1]$, where $\mathscr{A}$ is a σ-algebra of subsets of a given set (space of random events) and $\mathscr{B}$ is a non-empty subfamily of $\mathscr{A}$ (space of conditions), which is assumed to be a probability measure on $\mathscr{A}$ for every $B\in\mathscr{B}$ and to satisfy a natural condition of compatibility of the family of these measures.

This approach makes thus a generalization of the classical Kolmogorov approach. Moreover, Rényi's theory can be applied in various branches of mathematics and physics when Kolmogorov's approach fails or leads to complications (cf. [3]). In all such situations, calculations lead to unbounded measures of probability (in particular, to the uniform distribution on the whole real line), which do not make sense in the classical probability theory but can be defined in Rényi's approach (see [4], p. 245-254).

The aim of this paper, which can be treated as an expansion of some ideas and results of [3], is to discuss various types of extensions of Rényi's conditional probability spaces. More precisely, given a σ-algebra $\mathscr{A}$ of random events and a space $\mathscr{B}$ of conditions, we shall be interested in extending the family $\mathscr{B}$ by joining to it elements of $\mathscr{A}$ and in defining conditional probability $P(A\mid B)$ for $A\in\mathscr{A}$ and for B belonging to the extension of $\mathscr{B}$.

The following three types of extensions will be considered by adding to $\mathscr{B}$, respectively: 1° subsets of sets belonging to $\mathscr{B}$, 2° intersections of decreasing sequences in $\mathscr{B}$, 3° unions of increasing sequences in $\mathscr{B}$, under some necessary restrictions on sets which are to be joined to $\mathscr{B}$, resulting from properties of Rényi spaces.

We shall consider extensions of $\mathscr{B}$ by an arbitrary family of elements of $\mathscr{A}$ of a given type (with the restrictions mentioned above). In Sections 3-5, we give characterizations of Rényi spaces admitting such extensions. In particular, we obtain characterizations of extensions by one fixed element of $\mathscr{A}$ and of the maximal extensions by all possible elements of $\mathscr{A}$ of a given type.

The one-element extensions of types 1° and 3° are considered in [3]. However, the proof of the respective theorem for the extension 3° is not correct. We provide a new proof of this result (cf. Theorem 5.1 and Corollary 5.1).

In the case of the maximal extensions 1°-3°, we shall also give (in Sections 3-5) sufficient conditions for a given operation of extension to be idempotent.

It is interesting that the equivalent conditions $\mathrm{IV}_2^{(2)}$ and $\mathrm{IV}_3^{(2)}$, considered in [1] for different reasons, guarantee the feasibility of all the types of extensions as well as the idempotency of the maximal extension of type 1°. The maximal extensions 2° and 3° are idempotent under additional conditions, but in general only the ω_1-th iterations of extensions 2° and 3° lead to spaces of conditions closed on taking limits of decreasing (increasing) sequences of sets. It is worth noting that condition $\mathrm{IV}_1^{(2)}$, the weakest among conditions considered in [1], is not sufficient for any of the extensions 1°-3° to be feasible.

Note that if we perform the maximal extensions 1° and 3°, the second one ω_1 times, then the obtained space of conditions is closed with respect to all the operations 1°-3°, so this property of the space of conditions can be postulated in a given Rényi space in case of need (see [5] and [6]).

1. Axioms and their consequences. In the sequel, the set of all positive integers will be denoted by N.

By a *Rényi space* we mean a system $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$, where Ω is an arbitrary set (*space of elementary events*), $\mathscr{A}$ is a σ-algebra of subsets of Ω (*space of random events*), $\mathscr{B}$ is a non-empty subfamily of $\mathscr{A}$ (*space of conditions*), and P is a non-negative function on $\mathscr{A} \times \mathscr{B}$ (*conditional probability*) such that

(I) $P(B\,|\,B) = 1$ *for every* $B \in \mathscr{B}$;

(II) $P(\bigcup_{i=1}^{\infty} A_i\,|\,B) = \sum_{i=1}^{\infty} P(A_i\,|\,B)$ *for any disjoint sets* $A_i \in \mathscr{A}$ $(i \in N)$ *and every* $B \in \mathscr{B}$;

(III) *if* $A \in \mathscr{A}$, B, $B' \in \mathscr{B}$, $B \subset B'$, *and* $P(B\,|\,B') > 0$, *then*

$$P(A\,|\,B) = \frac{P(A \cap B\,|\,B')}{P(B\,|\,B')}$$

(see [3], p. 289, 296; [4], p. 70).

Let $\mathscr{K} = [\Omega, \mathscr{A}, P_0]$ be a Kolmogorov space (i.e., Ω is a given set, $\mathscr{A}$ is a σ-algebra of its subsets, and P_0 is a probability measure on $\mathscr{A}$). The system $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$, where $\mathscr{B}$ is the family of all sets $B \in \mathscr{A}$ such that

$P_0(B) > 0$ and $P(A \mid B) = P_0(A \cap B)/P_0(B)$ for $A \in \mathscr{A}$ and $B \in \mathscr{B}$, is a Rényi space.

One can easily check the following

THEOREM 1.1 (cf. [1], p. 338-339). *Let* $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ *fulfil axioms* (I) *and* (II). *Then axiom* (III) *is equivalent to the following two conditions*:

$$(\mathrm{III}_1) \qquad P(A \mid B) = P(A \cap B \mid B) \quad \textit{for } A = \mathscr{A},\ B \in \mathscr{B}$$

and

$$(\mathrm{III}_2) \quad P(A \mid B) \cdot P(B \mid B') = P(A \mid B') \quad \textit{if } A \subset B \subset B' \ (A \in \mathscr{A};\ B, B' \in \mathscr{B}).$$

Some results of this paper can be formulated for systems satisfying (I), (II), (III_1), more general than Rényi spaces, but this will not be marked in the paper.

In the sequel, the letters A and B with possible indices will always denote elements of the families $\mathscr{A}$ and $\mathscr{B}$, respectively, in a given Rényi space $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$. For two Rényi spaces $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ and $\tilde{\mathscr{R}} = [\Omega, \mathscr{A}, \tilde{\mathscr{B}}, \tilde{P}]$, we shall write $\mathscr{R} \subset \tilde{\mathscr{R}}$ if $\mathscr{B} \subset \tilde{\mathscr{B}}$ and $P = \tilde{P}$ on $\mathscr{A} \times \mathscr{B}$ and $\mathscr{R} = \tilde{\mathscr{R}}$ if $\mathscr{B} = \tilde{\mathscr{B}}$ and $P = \tilde{P}$.

The following properties of Rényi spaces are evident:

THEOREM 1.2 (cf. [3], p. 290-291; [4], p. 71). *Let* $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ *be a Rényi space. Then*

$$(1.1) \qquad P(\emptyset \mid B) = 0,$$

$$(1.2) \qquad P(\Omega \mid B) = 1,$$

$$(1.3) \qquad P(A \mid B) \leqslant 1,$$

$$(1.4) \qquad P(A \mid B) = 0 \quad \textit{if } A \cap B = \emptyset,$$

$$(1.5) \qquad P(A \mid B) \leqslant P(A' \mid B) \quad \textit{if } A \subset A',$$

$$(1.6) \qquad P(A \mid B') \leqslant P(A \mid B) \quad \textit{if } A \subset B \subset B',$$

$$(1.7) \qquad \emptyset \notin \mathscr{B}$$

for arbitrary elements of $\mathscr{A}$ *and* $\mathscr{B}$, *respectively.*

By (II), (1.1), (1.2) and (III), (III_1), we have

THEOREM 1.3 (cf. [3], p. 289-290; [4], p. 71). *Let* $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ *be a Rényi space and let* $P_0(A) = P(A \mid B_0)$ *for fixed* $B_0 \in \mathscr{B}$. *Then* $\mathscr{K} = [\Omega, \mathscr{A}, P_0]$ *is a Kolmogorov space. Now, if* $B \in \mathscr{B}$, $P_0(B) > 0$, *and* $B \cap B_0 \in \mathscr{B}$, *then letting* $\tilde{P}(A \mid B) = P_0(A \cap B)/P_0(B)$ *we have* $\tilde{P}(A \mid B) = P(A \mid B \cap B_0)$.

Note that there exist Rényi spaces which are not Kolmogorov ones (see [3], p. 304-310; [4], p. 72-73).

THEOREM 1.4 (cf. [3], p. 291). *Let* $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ *be a Rényi space* *and let* $A^1, A^2 \in \mathscr{A}$ *and* $B^1, B^2 \in \mathscr{B}$. *If*

$$A^1, A^2 \subset B^1 \cap B^2 \tag{1.8}$$

and

$$B^1 \cap B^2 \in \mathscr{B}, \tag{1.9}$$

then

$$P(A^1 \mid B^1) \cdot P(A^2 \mid B^2) = P(A^2 \mid B^1) \cdot P(A^1 \mid B^2). \tag{1.10}$$

If (1.8) *and* (1.9) *hold and*

$$P(A^2 \mid B^1) \cdot P(A^2 \mid B^2) > 0, \tag{1.11}$$

then

$$\frac{P(A^1 \mid B^1)}{P(A^2 \mid B^1)} = \frac{P(A^1 \mid B^2)}{P(A^2 \mid B^2)}. \tag{1.12}$$

In connection with (1.11), consider the inequalities

$$P(A^1 \mid B^1) \cdot P(A^1 \mid B^2) > 0, \tag{1.13}$$

$$P(B^1 \mid B^2) \cdot P(B^2 \mid B^1) > 0. \tag{1.14}$$

THEOREM 1.5. *In an arbitrary Rényi space, the following conditions are equivalent*:

(i) (1.8) $\wedge$ (1.11) $\Rightarrow$ (1.12);
(ii) (1.8) $\wedge$ [(1.11) $\vee$ (1.13)] $\Rightarrow$ (1.10);
(iii) (1.8) $\wedge$ (1.14) $\Rightarrow$ (1.10);
(iv) (1.8) $\Rightarrow$ (1.10).

Proof. The implications (iv) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i) are obvious.

To show that (iii) $\Rightarrow$ (ii), suppose (1.8) and, for instance, (1.11) hold true. But then, in view of (1.5) and (III_1), we have (1.14), so (1.10) follows by (iii).

Finally, in order to prove (i) $\Rightarrow$ (iv), assume (1.8) holds true.

If $P(B^1 \mid B^2) = 0$, then $P(A^1 \mid B^2) = P(A^2 \mid B^2) = 0$ by (1.8), (1.5), and (III_1). Consequently, (1.10) holds. Similar arguments work in case $P(B^2 \mid B^1) = 0$.

Therefore, we can assume (1.14) holds. Putting $A^2 = B^1 \cap B^2$ in (1.11), we get just (1.14), and thus (i) yields (1.12) for $A^2 = B^1 \cap B^2$. Hence the equivalence

$$P(A \mid B^1) = 0 \Leftrightarrow P(A \mid B^2) = 0 \tag{1.15}$$

holds for $A \subset B^1 \cap B^2$. But (1.15) and (i) imply (1.10), as desired.

Let us denote by (IV) the simplest from among conditions (i)-(iv) and let us treat it as an additional axiom:

(IV) $P(A^1 | B^1) \cdot P(A^2 | B^2) = P(A^2 | B^1) \cdot P(A^1 | B^2)$ *for* $A^1, A^2 \in \mathscr{A}$ *and* $B^1, B^2 \in \mathscr{B}$ *such that* $A^1, A^2 \subset B^1 \cap B^2$.

Remark 1.1. Condition (IV), stronger than the implication (1.8) $\wedge$ (1.9) $\wedge$ (1.11) $\Rightarrow$ (1.12) in Theorem 1.4, does not hold in Rényi spaces in general (see Example 3.1). In the form (i), this condition is adopted as an additional axiom in [3] (p. 301). The equivalence of (iii) and (iv) is shown in [1] (p. 354), where those conditions, denoted by $IV_2^{(2)}$ and $IV_3^{(2)}$, are discussed in connection with problems of representations of Rényi spaces by families of measures. Among various conditions considered in [1] (see p. 354), the only weaker than $IV_2^{(2)}$ and $IV_3^{(2)}$ is the following one:

$IV_1^{(2)}$. $P(A^1 | B^1) \cdot P(A^2 | B^2) = P(A^2 | B^1) \cdot P(A^1 | B^2)$ *for* $A^1, A^2 \in \mathscr{A}$ *and* $B^1, B^2 \in \mathscr{B}$, $A^1, A^2 \subset B^1 \cap B^2$ *provided all the factors are positive.*

By (1.7), it is impossible to desire $B^1 \cap B^2 \in \mathscr{B}$ to hold for every $B^1, B^2 \in \mathscr{B}$. However, the following condition seems to be quite natural:

(IV') *If* $B^1, B^2 \in \mathscr{B}$ *and* $P(B^1 | B^2) + P(B^2 | B^1) > 0$, *then* $B^1 \cap B^2 \in \mathscr{B}$.

COROLLARY 1.1. *In every Rényi space,* (IV') *implies* (IV).

Proof. Suppose that (IV') holds and let $A^1, A^2 \in \mathscr{A}$, $B^1, B^2 \in \mathscr{B}$, $A^1 \cup A^2 \subset B^1 \cap B^2$, and $P(B^1 | B^2) \cdot P(B^2 | B^1) > 0$. The last inequality implies $P(B^1 | B^2) + P(B^2 | B^1) > 0$, so $B^1 \cap B^2 \in \mathscr{B}$ in view of (IV'). By Theorem 1.4, we derive (1.10), i.e., the implication (IV') $\Rightarrow$ (iii) is shown. This completes the proof, because (iii) $\Leftrightarrow$ (IV) by Theorem 1.5.

The implication (IV) $\Rightarrow$ (IV') is not true (see Example 4.3).

2. Continuity of conditional probability. Let $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ be a fixed Rényi space.

Since $P(\cdot | B)$ is a probability measure on $\mathscr{A}$ for each fixed $B \in \mathscr{B}$, we have

$$\lim_{i \to \infty} P(A_i | B) = P(A | B) \tag{2.1}$$

for $B \in \mathscr{B}$ and for each monotone (increasing or decreasing) sequence $\{A_i\}$, $A_i \in \mathscr{A}$, with the limit (union or intersection, respectively) equal to A.

The continuity of P with respect to the second variable is considered in [3] (p. 302-303), but the cases of increasing and decreasing sequences are formulated asymmetrically and, in the second case, in a somewhat complicated form (cf. Corollaries 2.1 and 2.2).

We give a more general and symmetric formulation:

THEOREM 2.1. *Let* $\{A_i\}$ *and* $\{B_i\}$ *be increasing (decreasing) sequences of elements of* $\mathscr{A}$ *and* $\mathscr{B}$, *respectively, and let* A *and* B *be their limits, respectively.*

Suppose that $B \in \mathscr{B}$ *and, in case of decreasing* $\{B_i\}$, *assume additionally that*

$$P(B \mid B_{i_0}) > 0 \tag{2.2}$$

for some $i_0 \in N$. *Then*

$$\begin{aligned} \lim_{i\to\infty} \lim_{j\to\infty} P(A_i \mid B_j) &= \lim_{j\to\infty} \lim_{i\to\infty} P(A_i \mid B_j) \\ &= \lim_{i,j\to\infty} P(A_i \mid B_j) = \lim_{i\to\infty} P(A_i \mid B_i) = P(A \mid B). \end{aligned} \tag{2.3}$$

Proof. First of all notice that (2.1) implies

$$\begin{aligned} \lim_{i\to\infty} \lim_{j\to\infty} P(A_i \cap \tilde{A}_j \mid B) &= \lim_{j\to\infty} \lim_{i\to\infty} P(A_i \cap \tilde{A}_j \mid B) \\ &= \lim_{i,j\to\infty} P(A_i \cap \tilde{A}_j \mid B) = \lim_{i\to\infty} P(A_i \cap \tilde{A}_i \mid B) = P(A \cap \tilde{A} \mid B) \end{aligned} \tag{2.4}$$

for $B \in \mathscr{B}$ and arbitrary increasing (decreasing) sequences $\{A_i\}$ and $\{\tilde{A}_i\}$ of elements of $\mathscr{A}$ with limits A and $\tilde{A}$, respectively.

By (I), (2.1), and (2.2), we have 1° $P(B_j \mid B) > 0$ (with $B = \bigcup_{i=1}^{\infty} B_i$) if the sequence $\{B_i\}$ is increasing, and 2° $P(B_j \mid B_{i_0}) > 0$ if $\{B_i\}$ is decreasing for sufficiently large j.

By (III), in cases 1° and 2° we have

$$P(A_i \mid B_j) = \frac{P(A_i \cap B_j \mid B)}{P(B_j \mid B)} \quad \text{and} \quad P(A_i \mid B_j) = \frac{P(A_i \cap B_j \mid B_{i_0})}{P(B_j \mid B_{i_0})},$$

respectively, and thus (2.3) follows by (2.4).

Corollary 2.1 (cf. [3], p. 302). *Let* $A \in \mathscr{A}$ *and let* $\{B_i\}$ *be a monotone sequence in* $\mathscr{B}$ *such that its limit* B *is also in* $\mathscr{B}$. *Suppose additionally* (2.2) *holds for some* $i_0 \in N$ *in case of decreasing* $\{B_i\}$. *Then*

$$\lim_{i\to\infty} P(A \mid B_i) = P(A \mid B). \tag{2.5}$$

Proof. It suffices to put $A_i = A$ in Theorem 1.1.

Corollary 2.2 (see [3], p. 302). *Let* $A \in \mathscr{A}$, $B_i \supset B_{i+1}$ $(i \in N)$, $\bigcap_{i=1}^{\infty} B_i = B$, *and let* $\tilde{B} \in \mathscr{B}$ *be a set such that* $B \cap \tilde{B}$ *and* $B_i \cap \tilde{B}$ *are in* $\mathscr{B}$ *for* $i \in N$. *If* $P(B \mid \tilde{B}) > 0$, *then*

$$\lim_{i\to\infty} P(A \mid B_i \cap \tilde{B}) = P(A \mid B \cap \tilde{B}).$$

Proof. The sets $\bar{B}_i = B_i \cap \tilde{B}$ and $\bar{B} = B \cap \tilde{B}$ are in $\mathscr{B}$ and satisfy the assumptions of Corollary 2.1, because

$$P(\bar{B} \mid \bar{B}_i) \geqslant P(B \cap \tilde{B} \mid \tilde{B}) = P(B \mid \tilde{B}) > 0$$

for each $i \in N$ in view of (1.6) and (III_1). Thus the assertion follows by (2.5).

Remark 2.1. For increasing sequences $\{B_i\}$, $B_i \in \mathscr{B}$, with $B = \bigcup_{i=1}^{\infty} B_i \in \mathscr{B}$, we have

$$P(B_{i_0} \mid B) = \prod_{i=i_0}^{\infty} P(B_i \mid B_{i+1}) > 0 \tag{2.6}$$

for sufficiently large $i_0 \in N$. That means the condition $P(B_{i_0} \mid B) > 0$ or, in other words, the condition

$$\prod_{i=i_0}^{\infty} P(B_i \mid B_{i+1}) > 0 \tag{2.7}$$

is necessary for (2.5), and the more so for (2.3), to hold.

In fact, by induction, (III_2) yields

$$P(B_k \mid B_n) = \prod_{i=k}^{n-1} P(B_i \mid B_{i+1})$$

for every k, $n \in N$ and, letting $n \to \infty$, by (2.5) we get

$$P(B_k \mid B) = \prod_{i=k}^{\infty} P(B_i \mid B_{i+1}).$$

Hence (2.6) follows by (2.1) and (I).

In a similar way, one can show for decreasing sequences $\{B_i\}$, $B_i \in \mathscr{B}$, with $B = \bigcap_{i=1}^{\infty} B_i \in \mathscr{B}$ that (2.2) is equivalent to the condition

$$\prod_{i=i_0}^{\infty} P(B_{i+1} \mid B_i) > 0. \tag{2.8}$$

Note that condition (2.6) is necessary for increasing sequences to satisfy identity (2.1) and the equivalent conditions (2.2) and (2.8) are necessary for decreasing sequences to fulfil (2.5). In fact, suppose, on the contrary, that $P(B_i \mid B) = 0$ (or $P(B \mid B_i) = 0$) for all $i \in N$ if $\{B_i\}$ is an increasing (decreasing) sequence. Then (2.1) (or (2.5), respectively) results in $P(B \mid B) = 0$, which contradicts (I).

There is, however, an essential difference between the two cases. Condition (2.6) for increasing sequences is always valid, while conditions (2.2) and (2.8) need not be satisfied. On the other hand, conditions (2.2) and (2.8) cannot be omitted in Theorem 2.1 and Corollary 2.1 as the following example shows:

Example 2.1. Let $\Omega = (0, 1)$, $\mathscr{A}$ be the family of all Borel subsets of (0, 1), and $\mathscr{B}$ consist of the sets $B = (0, 1/2]$ and $B_i = (0, 1/2 + 1/2i)$ for $i \in N$. We define

$$P(A \mid B) = \frac{|A \cap B|}{|B|} \quad \text{and} \quad P(A \mid B_i) = \frac{|A \cap B_i \cap (\Omega \backslash B)|}{|B_i \cap (\Omega \backslash B)|}$$

for $A \in \mathscr{A}$ and $i \in N$. Axioms (I)-(IV) and (IV′) are satisfied in the space $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$.

We have $B_i \supset B_{i+1}$, $\bigcap_{i=1}^{\infty} B_i = B$, and B, $B_i \in \mathscr{B}$ $(i \in N)$, but $P(B \mid B_i) = 0$ for all $i \in N$.

3. Subsets. In a given Rényi space, we are going to study extensions of the family $\mathscr{B}$ of conditions by joining to it some random events belonging to the σ-algebra $\mathscr{A}$. Let us start from considering subsets of the sets belonging to $\mathscr{B}$.

THEOREM 3.1. *Let $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ be a Rényi space and fix $B_o \in \mathscr{A}$. Assume that there exists $B^1 \in \mathscr{B}$ such that*

$$B_o \subset B^1 \quad \textit{and} \quad P(B_o \mid B^1) > 0 \tag{3.1}$$

and the following condition is satisfied:

(α) *For every $B^2 \in \mathscr{B}$ such that $B^2 \supset B_o$, $P(B_o \mid B^2) > 0$ and for arbitrary A^1, $A^2 \in \mathscr{A}$ such that A^1, $A^2 \subset B_o$ identity* (1.10) *holds.*

Then the system $\mathscr{R}_o = [\Omega, \mathscr{A}, \mathscr{B}_o, P_o]$, where $\mathscr{B}_o = \mathscr{B} \cup \{B_o\}$, $P_o = P$ on $\mathscr{A} \times \mathscr{B}$, and

$$P_o(A \mid B_o) = \frac{P(A \cap B_o \mid B^1)}{P(B_o \mid B^1)} \quad (A \in \mathscr{A}), \tag{3.2}$$

is a Rényi space and definition (3.2) *is consistent, i.e., does not depend on the choice of a set B^1 satisfying* (3.1).

Proof (cf. [3], p. 297-298). It is easy to see from the definition of P_o that $\mathscr{R}_o$ satisfies axioms (I), (II), and axiom (III) in the cases $B = B' = B_o$ and $B \in \mathscr{B}$, $B' = B_o$ (i.e., $B \subset B_o$, $P_o(B \mid B_o) > 0$).

It remains to consider the case $B = B_o$, $B' \in \mathscr{B}$ (i.e., $B_o \subset B'$, $P(B_o \mid B') > 0$). But then we have $A \cap B_o \subset B_o \subset B^1 \cap B'$ for every $A \in \mathscr{A}$ and condition (α) yields

$$\frac{P(A \cap B_o \mid B^1)}{P(B_o \mid B^1)} = \frac{P(A \cap B_o \mid B')}{P(B_o \mid B')}. \tag{3.3}$$

It suffices now to use the definition of P_o in (3.3).

Condition (α) guarantees also the consistency of definition (3.2).

COROLLARY 3.1 (see [3], Theorem 10). *Let $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ be a Rényi space with a fixed $B_o \in \mathscr{A}$. Suppose that there exists $B^1 \in \mathscr{B}$ satisfying* (3.1) *and the following condition is fulfilled:*

(α') *$B^1 \cap B^2 \in \mathscr{B}$ for each $B^2 \in \mathscr{B}$ such that $B_o \subset B^2$ and $P(B_o \mid B^2) > 0$.*

Then $\mathscr{R}_o$ defined in Theorem 3.1 *is a Rényi space and definition* (3.2) *is consistent.*

Proof. Since, by Theorem 1.4, condition (α') implies (α), the assertion follows from Theorem 3.1.

Remark 3.1. In the case $B_0 \in \mathscr{B}$, the statements of Theorem 3.1 and Corollary 3.1 follow immediately by (III).

Given a Rényi space $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$, we denote in the sequel by $\mathscr{R}_0 = [\Omega, \mathscr{A}, \mathscr{B}_0, P_0]$ a fixed system, where $\mathscr{B}_0$ is an arbitrary family of sets $B_0 \in \mathscr{A}$ for which there exist sets $B^1 \in \mathscr{B}$ satisfying (3.1) and $P_0(A \mid B_0)$ is defined by formula (3.2) for $A \in \mathscr{A}$ and $B_0 \in \mathscr{B}_0$. In particular, if the extended family of conditions $\mathscr{B}_0$ contains all sets B_0 as above, the respective system will be denoted by $\mathscr{R}^0 = [\Omega, \mathscr{A}, \mathscr{B}^0, P^0]$.

Now, consider condition (α) for all $B_0 \in \mathscr{B}_0$ and denote it by (α_0):

(α_0) *For every* $B_0 \in \mathscr{B}_0$ *and* $B^1, B^2 \in \mathscr{B}$ *such that* $B^1, B^2 \supset B_0$ *and* $P(B_0 \mid B^i) > 0$ $(i = 1, 2)$ *and for every* $A^1, A^2 \in \mathscr{A}$ *such that* $A^1, A^2 \subset B_0$ *identity* (1.10) *holds.*

Remark 3.2. In the case $\mathscr{R}_0 = \mathscr{R}^0$, the above condition turns into condition (IV). In fact, suppose that $B^1, B^2 \in \mathscr{B}$ and $P(B^1 \mid B^2) \cdot P(B^2 \mid B^1) > 0$. By (III$_1$) we have $P(B^1 \cap B^2 \mid B^i) > 0$ $(i = 1, 2)$, so $B^1 \cap B^2 \in \mathscr{B}^0$. Thus condition ($\alpha_0$) (for $\mathscr{R}_0 = \mathscr{R}^0$) yields (1.10). This means that condition (iii) from Section 1 holds. But (iii), in view of Theorem 1.5, is equivalent to (IV), so condition (α_0) for $\mathscr{R}_0 = \mathscr{R}^0$ implies (IV). The converse implication is obvious.

THEOREM 3.2. *Let $\mathscr{R}$ be a Rényi space. The system $\mathscr{R}_0$ is a Rényi space iff $\mathscr{R}$ satisfies condition* (α_0). *In particular, $\mathscr{R}^0$ is a Rényi space iff $\mathscr{R}$ satisfies* (IV). *If $\mathscr{R}$ fulfils* (IV), *then $\mathscr{R}^0$ also fulfils* (IV) *and, moreover,*

$$\mathscr{R} \subset \mathscr{R}_0 \subset \mathscr{R}^0 \tag{3.4}$$

and

$$\mathscr{R}^{00} = \mathscr{R}^0. \tag{3.5}$$

Proof. Suppose that $\mathscr{R}$ satisfies (α_0) and let B_0 and B_0' be two arbitrary elements of $\mathscr{B}_0$. By Theorem 3.1, the extended system $\mathscr{R}' = [\Omega, \mathscr{A}, \mathscr{B}', P']$, where $\mathscr{B}' = \mathscr{B} \cup \{B_0'\}$, $P' = P$ on $\mathscr{A} \times \mathscr{B}$, and

$$P'(A \mid B_0') = \frac{P(A \cap B_0' \mid B')}{P(B_0' \mid B')}$$

for some $B' \in \mathscr{B}$ such that $B_0' \subset B'$ and $P(B_0' \mid B') > 0$.

Note that condition (α) is satisfied for the system $\mathscr{R}'$. In fact, if $A^1, A^2 \in \mathscr{A}$, $B^1, B^2 \in \mathscr{B}$, and $A^1 \cap A^2 \subset B_0 \subset B^1 \cap B^2$, then (1.10) holds in view of condition (α_0) assumed for $\mathscr{R}$. The case where one of the sets B^1, B^2 or both of them coincide with B_0' reduces to the previous one, according to the definition of P'.

Therefore, we can apply again Theorem 3.1, now for the system $\mathscr{R}'$. As a consequence, axioms (I)-(III) are satisfied for arbitrary elements of $\mathscr{A}$ and $\mathscr{B} \cup \{B_0, B_0'\}$, respectively.

Since B_o and B'_o were chosen arbitrarily in $\mathscr{B}_o$, we have proved that $\mathscr{R}_o$ is a Rényi space.

Suppose now that $\mathscr{R}_o$ is a Rényi space and let $B_o \in \mathscr{B}_o$; $B^1, B^2 \in \mathscr{B}$; $B^1, B^2 \supset B_o$, $P(B_o | B^i) > 0$ $(i = 1, 2)$. Since (III) holds for $\mathscr{R}^o$, we have

$$\frac{P(A | B^1)}{P(B_o | B^1)} = P_o(A | B_o) = \frac{P(A | B^2)}{P(B_o | B^2)}$$

for every set $A \in \mathscr{A}$, $A \subset B_o$. Hence (1.10) holds for $A^1, A^2 \in \mathscr{A}$ such that $A^1, A^2 \subset B_o$. This means that condition (α_o) holds and the first part of the theorem is proved.

The statement in the case $\mathscr{R}_o = \mathscr{R}^o$ follows according to Remark 3.2.

Now, assume that the Rényi space $\mathscr{R}$ satisfies axiom (IV) and let $B_o^1, B_o^2 \in \mathscr{B}^o$, i.e., $B_o^i \subset B^i \in \mathscr{B}$ and $P(B_o^i | B^i) > 0$ $(i = 1, 2)$. By the definition of P^o and by (IV), assumed for $\mathscr{R}$, we have

$$\begin{aligned} P^o(A^1 | B_o^1) \cdot P^o(A^2 | B_o^2) &= \frac{P(A^1 | B^1) \cdot P(A^2 | B^2)}{P(B_o^1 | B^1) \cdot P(B_o^2 | B^2)} \\ &= \frac{P(A^2 | B^1) \cdot P(A^1 | B^2)}{P(B_o^1 | B^1) \cdot P(B_o^2 | B^2)} = P^o(A^2 | B_o^1) \cdot P^o(A^1 | B_o^2) \end{aligned}$$

for $A^1, A^2 \in \mathscr{A}$, $A^1, A^2 \subset B_o^1 \cap B_o^2$, i.e., $\mathscr{R}^o$ fulfils (IV).

Relation (3.4) is guaranteed by axiom (III), assumed for $\mathscr{R}$.

Since for two arbitrary Rényi spaces $\mathscr{R}_1$ and $\mathscr{R}_2$ the relation $\mathscr{R}_1 \subset \mathscr{R}_2$ implies $\mathscr{R}_1^o \subset \mathscr{R}_2^o$, (3.4) yields $\mathscr{R}^o \subset \mathscr{R}^{oo}$.

Assume now that $B_{oo} \in \mathscr{B}^{oo}$, i.e., there exist sets B_o and B from $\mathscr{B}^o$ and $\mathscr{B}$, respectively, such that $B_{oo} \subset B_o \subset B$ and $P^o(B_{oo} | B) > 0$, $P(B_o | B) > 0$. Moreover,

$$P^o(A | B_o) = \frac{P(A \cap B_o | B)}{P(B_o | B)}$$

for every $A \in \mathscr{A}$, and hence

$$P(B_{oo} | B) = P^o(B_{oo} | B_o) \cdot P(B_o | B) > 0,$$

i.e., $B_{oo} \in \mathscr{B}^o$, and the inclusion $\mathscr{B}^{oo} \subset \mathscr{B}^o$ is proved.

Consequently, $\mathscr{B}^o = \mathscr{B}^{oo}$ and $P^o = P^{oo}$ on $\mathscr{A} \times \mathscr{B}^o = \mathscr{A} \times \mathscr{B}^{oo}$ since $\mathscr{R}^o \subset \mathscr{R}^{oo}$, so (3.5) holds and the proof is complete.

As a consequence of Theorem 3.2 and Corollary 1.1, we get

THEOREM 3.3. *If a Rényi space $\mathscr{R}$ satisfies condition* (IV′), *then $\mathscr{R}_o$ and $\mathscr{R}^o$ are Rényi spaces and $\mathscr{R}^o$ satisfies* (IV′) *and* (3.5).

Proof. We need only to prove that $\mathscr{R}^o$ fulfils (IV′).

Let $B^1, B^2 \in \mathscr{B}^o$ and suppose, for instance, that $P^o(B^1 | B^2) > 0$. Then there is a set $\tilde{B} \in \mathscr{B}$ such that $B^2 \subset \tilde{B}$, $P(B^2 \mid \tilde{B}) > 0$, and

$$P^{\circ}(B^1 \mid B^2) = \frac{P(B^1 \cap B^2 \mid \tilde{B})}{P(B^2 \mid \tilde{B})}.$$

Hence $P(B^1 \cap B^2 \mid \tilde{B}) > 0$, and this yields $B^1 \cap B^2 \in \mathscr{B}^{\circ}$, as desired.

The following example shows that conditions (α), (α'), ($\alpha_{\circ}$), (IV), and (IV′) in Theorems 3.1, 3.2, 3.3 and Corollary 3.1, respectively, are essential and that condition $\mathrm{IV}_1^{(2)}$ is not sufficient for $\mathscr{R}^{\circ}$ to be a Rényi space.

Example 3.1. Let $\Omega = \{1, 2, 3, 4\}$, $\mathscr{A} = 2^{\Omega}$, $\mathscr{B} = \{B^1, B^2\}$, where $B^1 = \{1, 2, 3\}$, $B^2 = \{2, 3, 4\}$, and let $P(A \mid B^i) = \delta_{i+1}(A)$ $(i = 1, 2;\ A \in \mathscr{A})$, where δ_c is a probability measure concentrated at the point c. Clearly, $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ is a Rényi space and $\mathrm{IV}_1^{(2)}$ holds. But

$$1 = P(A^1 \mid B^1) \cdot P(A^2 \mid B^2) \neq P(A^2 \mid B^1) \cdot P(A^1 \mid B^2) = 0$$

for $A^1 = \{2\}$, $A^2 = \{2, 3\}$, i.e., (IV) does not hold. Putting $B_{\circ} = \{2, 3\}$, we see that conditions (α), (α'), ($\alpha_{\circ}$) and (IV′) for $\mathscr{B}_{\circ} = \mathscr{B} \cup \{B_{\circ}\}$ and $\mathscr{B}^{\circ}$ consisting of all $A \in \mathscr{A}$ such that $A \cap \{2, 3\} = \emptyset$ are not fulfilled either.

Defining $P_{\circ} = P^{\circ} = P$ on $\mathscr{A} \times \mathscr{B}$ and

$$P_{\circ}(A \mid B_{\circ}) = P^{\circ}(A \mid B_{\circ}) = \frac{P(A \cap B_{\circ} \mid B^1)}{P(B_{\circ} \mid B^1)} \qquad (A \in \mathscr{A}),$$

we have

$$P_{\circ}(A \mid B_{\circ}) = P^{\circ}(A \mid B_{\circ}) \neq \frac{P_{\circ}(A \cap B_{\circ} \mid B^2)}{P_{\circ}(B_{\circ} \mid B^2)} = \frac{P^{\circ}(A \cap B_{\circ} \mid B^2)}{P^{\circ}(B_{\circ} \mid B^2)}$$

for $A = \{2\}$ or $A = \{3\}$, i.e., $\mathscr{R}_{\circ} = [\Omega, \mathscr{A}, \mathscr{B}_{\circ}, P_{\circ}]$ and $\mathscr{R}^{\circ} = [\Omega, \mathscr{A}, \mathscr{B}^{\circ}, P^{\circ}]$ are not Rényi spaces.

4. Intersections. Next, we shall extend a given Rényi space adding to the family of conditions $\mathscr{B}$ intersections $B_{\bullet} = \bigcap_{i=1}^{\infty} B_i$ of decreasing sequences $\{B_i\}$, $B_i \in \mathscr{B}$, and defining a conditional probability for the extended space of conditions by formula (2.5). This kind of extensions was not considered in [3].

Since from Remark 2.1 it follows that condition (2.8) is necessary for such extensions, we assume in this section that decreasing sequences $\{B_i\}$, $B_i \in \mathscr{B}$, satisfy the condition

$$\prod_{i=1}^{\infty} P(B_{i+1} \mid B_i) > 0. \tag{4.1}$$

THEOREM 4.1. *Let $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ be a Rényi space and let*

$$B_\bullet = \bigcap_{i=1}^{\infty} B_i \tag{4.2}$$

for a fixed decreasing sequence $\{B_i\}$, $B_i \in \mathscr{B}$, satisfying (4.1). *Suppose that*

(β) *for each $B \in \mathscr{B}$, $B \supset B_\bullet$ and for each $i \in N$ identity* (1.10) *holds with $A^1, A^2 \subset B_\bullet$ and $B^1 = B$, $B^2 = B_i$.*

Then the system $\mathscr{R}_\bullet = [\Omega, \mathscr{A}, \mathscr{B}_\bullet, P_\bullet]$, where $\mathscr{B}_\bullet = \mathscr{B} \cup \{B_\bullet\}$, $P_\bullet(A \mid B) = P(A \mid B)$ for $A \in \mathscr{A}$, $B \in \mathscr{B}$, and

$$P_\bullet(A \mid B_\bullet) = \lim_{i \to \infty} P(A \mid B_i) \quad (A \in \mathscr{A}), \tag{4.3}$$

is a Rényi space. Moreover, definition (4.3) *is consistent, i.e. does not depend on the choice of a decreasing sequence $\{B_i\}$, $B_i \in \mathscr{B}$, satisfying* (4.1) *and* (4.2).

Proof. Since (III_2) yields

$$P(B_n \mid B_k) = \prod_{i=k}^{n-1} P(B_{i+1} \mid B_i) \quad \text{for } n > k,$$

condition (4.1) implies $P(B_j \mid B_i) > 0$ for all $i, j \in N$ and, by (2.1), we have $P(B \mid B_i) > 0$ for $i \in N$. Thus (III) yields

$$P(A \mid B_j) = \frac{P(A \cap B_j \mid B_i)}{P(B_j \mid B_i)} \quad \text{for } j \geqslant i$$

and, letting $j \to \infty$, we get

$$P_\bullet(A \mid B_\bullet) = \frac{P(A \cap B_\bullet \mid B_i)}{P(B_\bullet \mid B_i)} \quad \text{for } i \in N. \tag{4.4}$$

The latter formula allows us to check easily that $\mathscr{R}_\bullet$ fulfils axioms (I), (II), and (III) in the cases $B = B' = B_\bullet$ and $B \in \mathscr{B}$, $B' = B_\bullet$. It remains to show that

$$P_\bullet(A \mid B_\bullet) = \frac{P(A \cap B_\bullet \mid B)}{P(B_\bullet \mid B)}$$

for every $B \in \mathscr{B}$ such that $P(B_\bullet \mid B) > 0$. But, by (4.4), it suffices to use (β) for $A^1 = A \cap B_\bullet$ and $A^2 = B_\bullet$.

Now, let $\{\bar{B}_i\}$, $\bar{B}_i \in \mathscr{B}$, be another decreasing sequence satisfying (4.1) and (4.2). Since $B_\bullet \subset \bar{B}_1$ and, by (4.1), $P(B_\bullet \mid \bar{B}_1) > 0$, we have

$$P_\bullet(A \mid B_\bullet) = \frac{P(A \cap B_\bullet \mid \bar{B}_1)}{P(B_\bullet \mid \bar{B}_1)} = \lim_{i \to \infty} \frac{P(A \cap \bar{B}_i \mid \bar{B}_1)}{P(\bar{B}_i \mid \bar{B}_1)} = \lim_{i \to \infty} P(A \mid \bar{B}_i)$$

by (III) and (2.1). The consistency of definition (4.3) is thus proved.

Remark 4.1. In the case where $B_\bullet \in \mathscr{B}$, the above statements are evident by Theorem 2.1.

Given a Rényi space $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$, fix now an arbitrary family $\mathscr{B}_\bullet$ of sets $B_\bullet$ of the form (4.2), where $\{B_i\}$, $B_i \in \mathscr{B}$, is a decreasing sequence satisfying (4.1), and define $P_\bullet(A \mid B_\bullet)$ for all $A \in \mathscr{A}$ and $B_\bullet \in \mathscr{B}_\bullet$ by (4.3). In the sequel, we apply the notation $\mathscr{R}_\bullet = [\Omega, \mathscr{A}, \mathscr{B}_\bullet, P_\bullet]$ for that fixed system.

In particular, if $\mathscr{B}_\bullet$ consists of all sets $B_\bullet$ defined above, we denote the respective system by $\mathscr{R}^\bullet = [\Omega, \mathscr{A}, \mathscr{B}^\bullet, P^\bullet]$.

Now, the following analogues of condition (β) will be considered:

($\beta_\bullet$) *For each $B_\bullet \in \mathscr{B}_\bullet$, for each $B \in \mathscr{B}$, $B \supset B_\bullet$, and for each $i \in N$ identity* (1.10) *holds with A^1, $A^2 \subset B_\bullet$ and $B^1 = B$, $B^2 = B_i$, where $\{B_i\}$, $B_i \in \mathscr{B}$, is a decreasing sequence satisfying* (4.1) *and* (4.2).

($\beta^\bullet$) *For each $B_\bullet$ of the form* (4.2), *for each $B \in \mathscr{B}$, $B \supset B_\bullet$, and for each $i \in N$ identity* (1.10) *holds with A^1, $A^2 \subset B_\bullet$ and $B^1 = B$, $B^2 = B_i$, where $\{B_i\}$, $B_i \in \mathscr{B}$, is a decreasing sequence satisfying* (4.1) *and* (4.2).

THEOREM 4.2. *Let $\mathscr{R}$ be a Rényi space. The system $\mathscr{R}_\bullet$ is a Rényi space iff $\mathscr{R}$ satisfies condition* ($\beta_\bullet$). *In particular, $\mathscr{R}^\bullet$ is a Rényi space iff $\mathscr{R}$ satisfies* ($\beta^\bullet$). *Moreover,*

$$\mathscr{R} \subset \mathscr{R}_\bullet \subset \mathscr{R}^\bullet \subset \mathscr{R}^0. \tag{4.5}$$

Proof. Suppose that $\mathscr{R}$ fulfils condition ($\beta_\bullet$) and let $B_\bullet$ and $B'_\bullet$ be two arbitrary elements of $\mathscr{B}_\bullet$. By Theorem 4.1, the system $\mathscr{R}' = [\Omega, \mathscr{A}, \mathscr{B}', P']$, where $\mathscr{B}' = \mathscr{B} \cup \{B'_\bullet\}$ and P' is defined as in Theorem 4.1, is a Rényi space. It is easy to see that the system $\mathscr{R}'$ also satisfies condition (β), so we can apply Theorem 4.1 once more. Consequently, the system extended by adding the sets $B_\bullet$ and $B'_\bullet$ to the family $\mathscr{B}$ is a Rényi space, so axioms (I)-(III) hold for arbitrary elements of $\mathscr{B} \cup \{B_\bullet, B'_\bullet\}$. Since $B_\bullet$ and $B'_\bullet$ were chosen arbitrarily, this means that $\mathscr{R}_\bullet$ is a Rényi space.

Assume now that $\mathscr{R}_\bullet$ is a Rényi space. Let $B_\bullet \in \mathscr{B}_\bullet$ and $\{B_i\}$, $B_i \in \mathscr{B}$, be a decreasing sequence satisfying (4.1) and (4.2). Moreover, let $B \in \mathscr{B}$, $B \supset B_\bullet$ and A^1, $A^2 \in \mathscr{A}$, A^1, $A^2 \subset B_\bullet$.

If $P(B_\bullet \mid B) > 0$, then

$$P_\bullet(A^j \mid B_\bullet) = \frac{P(A^j \mid B)}{P(B_\bullet \mid B)} = \frac{P(A^j \mid B_i)}{P(B_\bullet \mid B_i)} \tag{4.6}$$

for every $i \in N$ and $j = 1, 2$, by axiom (III) assumed for $\mathscr{R}_\bullet$. Hence

$$\begin{aligned} &P(A^1 \mid B) \cdot P(A^2 \mid B_i) \\ &\quad = P(B_\bullet \mid B) \cdot P(B_\bullet \mid B_i) \cdot P_\bullet(A^1 \mid B_\bullet) \cdot P_\bullet(A^2 \mid B_\bullet) = P(A^2 \mid B) \cdot P(A^1 \mid B_i) \end{aligned} \tag{4.7}$$

for each $i \in N$.

If $P(B_\bullet \mid B) = 0$, then, by (1.5), $P(A^j \mid B) = 0$ $(j = 1, 2)$, so (4.7) holds also in this case.

We have thus proved that $\mathscr{R}$ fulfils condition ($\beta_\bullet$) and the first part of the theorem as well as the statement concerning $\mathscr{R}^\bullet$ are proved.

The inclusions $\mathscr{R} \subset \mathscr{R}_\bullet \subset \mathscr{R}^\bullet$ are obvious. By (4.1), it is clear that $\mathscr{B}^\bullet \subset \mathscr{B}^\circ$ (see Remark 2.1). It remains to use formula (4.4) to see that $P^\bullet = P^\circ$ on $\mathscr{A} \times \mathscr{B}^\bullet$, so (4.5) is shown and the proof is completed.

Remark 4.2. It follows from the proofs of Theorems 4.1 and 4.2 that conditions a little weaker than ($\beta_\bullet$) and ($\beta^\bullet$) are sufficient for the proof that the systems $\mathscr{R}_\bullet$ and $\mathscr{R}^\bullet$, respectively, are Rényi spaces. Namely, the quantifier "for each $i \in N$" in these conditions can be replaced by the quantifier "for some $i \in N$". On the other hand, if $\mathscr{R}_\bullet$ or $\mathscr{R}^\bullet$ is a Rényi space, then the Rényi space $\mathscr{R}$ satisfies ($\beta_\bullet$) or ($\beta^\bullet$), respectively, in the form given above, i.e., for each $i \in N$.

It is also interesting that, in conditions (β), ($\beta_\bullet$), ($\beta^\bullet$), for a given $B_\bullet$ only one fixed sequence $\{B_i\}$, $B_i \in \mathscr{B}$, satisfying (4.1) and (4.2), is assumed to fulfil the respective assertion. One can easily derive from the proof of Theorem 4.2 that this assumption already implies the respective assertion for all such sequences $\{B_i\}$.

THEOREM 4.3. *If a Rényi space $\mathscr{R}$ satisfies* (IV), *then $\mathscr{R}^\bullet$ is also a Rényi space satisfying* (IV) *and*

$$\mathscr{R} \subset \mathscr{R}^\bullet \subset \mathscr{R}^\circ. \tag{4.8}$$

Proof. Since (IV) implies ($\beta^\bullet$), it suffices to show that (IV) holds in $\mathscr{R}^\bullet$, according to Theorem 4.2. But identity (1.10) assumed in condition ($\beta^\bullet$) implies

$$P(A^1 \mid B_\bullet) \cdot P(A^2 \mid B) = P(A^2 \mid B_\bullet) \cdot P(A^1 \mid B)$$

for arbitrary $B \in \mathscr{B}$, $B_\bullet \in \mathscr{B}^\bullet$, and $A^1, A^2 \in \mathscr{A}$ such that $A^1, A^2 \subset B \cap B_\bullet$, which yields (IV) in $\mathscr{R}^\bullet$.

THEOREM 4.4. *If a Rényi space $\mathscr{R}$ fulfils condition* (IV′), *then $\mathscr{R}^\bullet$ is a Rényi space, satisfying* (IV′), (4.8), *and*

$$\mathscr{R}^{\bullet\bullet} = \mathscr{R}^\bullet. \tag{4.9}$$

Proof. By Theorem 4.3, $\mathscr{R}^\bullet$ is a Rényi space fulfilling (4.8). It is not difficult to check that $\mathscr{R}^\bullet$ fulfils (IV′) (cf. the proof of Theorem 5.5), so it remains to show (4.9).

To prove the inclusion

$$\mathscr{B}^{\bullet\bullet} \subset \mathscr{B}^\bullet, \tag{4.10}$$

suppose that $B^{\bullet\bullet} \in \mathscr{B}^{\bullet\bullet}$, i.e., $B^{\bullet\bullet} = \bigcap_{i=1}^{\infty} B_i^\bullet$, where

$$B_i^\bullet \in B^\bullet;\ B_i^\bullet \supset B_{i+1}^\bullet \text{ for } i \in N \quad \text{and} \quad \prod_{j=1}^{\infty} P^\bullet(B_{j+1}^\bullet \mid B_j^\bullet) > 0. \tag{4.11}$$

Further, we have

$$B_i^\bullet = \bigcap_{k=1}^{\infty} B_{ik} \quad (i \in N),$$

where

$$(4.12)\qquad B_{ik}\in\mathscr{B},\quad B_{ik}\supset B_{i,k+1},\quad \text{and}\quad \prod_{j=1}^{\infty}P(B_{ij}\mid B_{i,j+1})>0$$

for $i,\ k\in N$.

By (4.11) and (4.12), we have

$$(4.13)\qquad P^{\bullet}(B^{\bullet\bullet}\mid B_i^{\bullet})>0\quad\text{and}\quad P(B_i^{\bullet}\mid B_{ik})>0$$

for $i,\ k\in N$.

We shall show, by induction, that

$$(4.14)\qquad \bar{B}_{ik}\in\mathscr{B}$$

for $i,\ k\in N$, where

$$\bar{B}_{ik}=\bigcap_{j=1}^{i}B_{jk}.$$

Let k be fixed. It is obvious that (4.14) holds for $i=1$.

Suppose that (4.14) holds for some $i\in N$. Since $B_i^{\bullet}\subset B_j^{\bullet}\subset B_{jk}$ for $j\leqslant i$, we have $B_i^{\bullet}\subset\bar{B}_{ik}\subset B_{ik}$, and thus

$$(4.15)\qquad P(B_i^{\bullet}\mid\bar{B}_{ik})\geqslant P(B_i^{\bullet}\mid B_{ik})>0$$

by (1.6) and (4.13). Further, we have

$$P(B_{i+1,k}\mid\bar{B}_{ik})\geqslant P(B_{i+1}^{\bullet}\mid\bar{B}_{ik})=P^{\bullet}(B_{i+1}^{\bullet}\mid B_i^{\bullet})\cdot P(B_i^{\bullet}\mid\bar{B}_{ik})>0$$

in view of (1.5), (III_2), (4.13), and (4.15). Now, axiom (IV′) yields

$$\bar{B}_{ik}\cap B_{i+1,k}\in\mathscr{B},$$

which completes the proof of (4.14).

Let $\tilde{B}_k=\bar{B}_{kk}$ for $k\in N$. Clearly,

$$(4.16)\qquad \tilde{B}_k\in\mathscr{B}\quad\text{and}\quad\tilde{B}_k\supset\tilde{B}_{k+1}$$

for $k\in N$, as a consequence of (4.12) and (4.14).

It is easy to see that

$$(4.17)\qquad \bigcap_{k=1}^{\infty}\tilde{B}_k=\bigcap_{i=1}^{\infty}B_i^{\bullet}=B^{\bullet\bullet}$$

and, moreover,

$$(4.18)\qquad P(B^{\bullet\bullet}\mid\tilde{B}_k)\geqslant P(B^{\bullet\bullet}\mid B_{kk})=P^{\bullet}(B^{\bullet\bullet}\mid B_k^{\bullet})\cdot P(B_k^{\bullet}\mid B_{kk})>0$$

by (1.6), (III_2), and (4.13).

Relations (4.16)–(4.18) imply $B^{\bullet\bullet}\in\mathscr{B}^{\bullet}$ (cf. Remark 2.1), and thus inclusion (4.10) is proved.

Hence (4.9) follows by (4.5).

Now, we are going to give an example that the assumptions (β) in Theorem 4.1, ($\beta_\bullet$) and ($\beta^\bullet$) in Theorem 4.2, (IV) in Theorem 4.3 and (IV′) in Theorem 4.4 cannot be omitted and, what is more, that condition (IV) cannot be replaced in Theorem 4.2 by condition $IV_1^{(2)}$ (see [1] or Section 1).

Example 4.1. Let $\Omega = [0, 1]$ and let $\mathscr{A}$ be the family of all Borel subsets of Ω. Putting $x = 1/3$, $y = 2/3$, $z = 1$ and $x_n = 1/3 - 1/3^n$ for $n \in N$, we define

$$B = \{x, y, z\}, \qquad B_n = \{x, y, x_k \colon k \geqslant n\},$$
$$\mathscr{B} = \{B, B_n \colon n \in N\}, \qquad B_\bullet = \bigcap_{i=1}^{\infty} B_i = \{x, y\}.$$

Moreover, let $P(A \mid B) = \delta_y(A)$ and

$$P(A \mid B_n) = [1 + 2^{-(n-1)}]^{-1} [\delta_x(A) + \sum_{i=n}^{\infty} 2^{-i} \delta_{x_i}(A)]$$

for $A \in \mathscr{A}$, where δ_c is a probability measure concentrated at c.

It is easy to see that $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ is a Rényi space and the sequence $\{B_i\}$ satisfies (4.1).

Equation (1.10) with $B^1 = B$ and $B^2 = B_i$ is fulfilled for A^1, $A^2 \in \mathscr{A}$, A^1, $A^2 \subset B \cap B_i$ under the assumption that all the factors are positive, but not in general, even if A^1, $A^2 \subset B_\bullet$ (e.g., this does not hold for $A^1 = \{x, y\}$ and $A^2 = \{x\}$). This means that condition $IV_1^{(2)}$ from [1] (see Section 1) is valid, but none of the conditions (β), ($\beta_\bullet$), ($\beta^\bullet$), (IV) holds. Clearly, (IV′) does not hold either.

Let

$$\mathscr{B}_\bullet = \mathscr{B} \cup \{B_\bullet\} (= \mathscr{B}^\bullet), \qquad P_\bullet = P^\bullet = P \text{ on } \mathscr{A} \times \mathscr{B},$$

and

$$P_\bullet(A \mid B_\bullet) = P^\bullet(A \mid B_\bullet) = \lim_{n \to \infty} P(A \mid B_n) = \delta_x(A) \quad (A \in \mathscr{A}).$$

We have $B_\bullet \subset B$, $P_\bullet(B_\bullet \mid B) = P^\bullet(B_\bullet \mid B) > 0$, and

$$P_\bullet(A \mid B_\bullet) = P^\bullet(A \mid B_\bullet) \neq \frac{P_\bullet(A \cap B_\bullet \mid B)}{P_\bullet(B_\bullet \mid B)} = \frac{P^\bullet(A \cap B_\bullet \mid B)}{P^\bullet(B_\bullet \mid B)}$$

for $A = \{x\}$ or for $A = \{y\}$. Hence $\mathscr{R}_\bullet = [\Omega, \mathscr{A}, \mathscr{B}_\bullet, P_\bullet]$ and $\mathscr{R}^\bullet = [\Omega, \mathscr{A}, \mathscr{B}^\bullet, P^\bullet]$ are not Rényi spaces.

As the following example shows, condition ($\beta^\bullet$) is essentially weaker than (IV) (cf. Theorems 4.2 and 4.3):

Example 4.2. Let $\Omega = [0, 1]^2$ and let $\mathscr{A}$ consist of all Borel subsets of Ω. Moreover, let $B_n = [0, 1/2] \times [0, 1/2 + 1/2n]$ for $n \in N$ and

$$B^1 = [0, 1] \times [0, 1/2], \qquad B^2 = [1/2, 1] \times [0, 1], \qquad C = [1/2, 1] \times [0, 1/4].$$

We adopt $\mathscr{B} = \{B^1, B^2, B_n: n \in N\}$ and

$$P(A \mid B^1) = \frac{|A \cap B^1|}{|B^1|}, \quad P(A \mid B^2) = \frac{|A \cap C|}{|C|}, \quad P(A \mid B_n) = \frac{|A \cap B_n|}{|B_n|}$$

for $n \in N$.

One can easily check that $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ is a Rényi space, the sequence $\{B_i\}$ fulfils (4.1), and ($\beta^\bullet$) holds.

However, condition (IV) is not valid because (1.10) does not hold, e.g., for

$$A^1 = [1/2, 1] \times [0, 1/2], \quad A^2 = [1/2, 1] \times [1/4, 1/2].$$

The following example will show that condition (IV) is not sufficient for a Rényi space to satisfy identity (4.9) and, in particular, that (IV) is an essentially weaker condition than (IV′) (cf. Theorems 4.3 and 4.4):

Example 4.3. Let $\Omega = [0, 1] \times [0, 1]$ and let $\mathscr{A}$ be the family of all Borel subsets of Ω. Moreover, we define the sets

$$B_{ij} = \left(\left[0, \frac{i+1}{3i}\right] \times \left[0, \frac{j+1}{3j}\right]\right) \cup \left(\left[\frac{1}{3}, \frac{i+1}{3i}\right] \times \left[\frac{j+1}{3j}, \frac{3j-1}{3j}\right]\right)$$

and the conditional probabilities

$$P(A \mid B_{ij}) = \frac{|A \cap B_{ij}|}{|B_{ij}|}$$

for $i, j \in N$ and $A \in \mathscr{A}$.

The system $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$, where $\mathscr{B} = \{B_{ij}: i, j \in N\}$ and P is defined as above, is a Rényi space fulfilling (IV). Of course, condition (IV′) is not satisfied.

It is easy to see that

$$\mathscr{B}^\bullet = \mathscr{B} \cup \{B_j: j \in N\},$$

where

$$B_j = \left[0, \frac{1}{3}\right] \times \left[0, \frac{j+1}{3}\right] \quad \text{and} \quad \mathscr{B}^{\bullet\bullet} = \mathscr{B} \cup \{B\},$$

where $B = [0, 1/3] \times [0, 1/3]$. This means that $\mathscr{B}^\bullet \neq \mathscr{B}^{\bullet\bullet}$.

One can consider the α-th iteration of the considered operation $^\bullet$ of the extension of a given Rényi space $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ satisfying (IV), where α is an arbitrary ordinal. Having defined $\mathscr{R}_0^\bullet = \mathscr{R}$ and the iterations $\mathscr{R}_\beta^\bullet = [\Omega, \mathscr{A}, \mathscr{B}_\beta^\bullet, P_\beta^\bullet]$ satisfying axioms (I)–(IV) for all $\beta < \alpha$ and such that

$$\mathscr{R}_\beta^\bullet \subset \mathscr{R}_{\beta'}^\bullet \quad \text{for } \beta < \beta' < \alpha, \tag{4.19}$$

we define

$$\mathscr{R}_\alpha^\bullet = \bigcup_{\beta<\alpha} (\mathscr{R}_\beta^\bullet)^\bullet,$$

i.e.,

$$\mathscr{B}_\alpha^\bullet = \bigcup_{\beta<\alpha} (\mathscr{B}_\beta^\bullet)^\bullet$$

and

$$(4.20)\qquad P_\alpha^\bullet(A\,|\,B) = P_\beta^\bullet(A\,|\,B) \quad \text{for } A\in\mathscr{A},\ B\in(\mathscr{B}_\beta^\bullet)^\bullet.$$

Definition (4.20) is correct because of (4.19).

THEOREM 4.5. *Given a Rényi space* $\mathscr{R}$ *with property* (IV), *the system* $\mathscr{R}_{\omega_1}^\bullet$ *is the smallest Rényi space containing* $\mathscr{R}$ *and closed with respect to the operation* $^\bullet$. *More precisely*:

(i) $\mathscr{R}_{\omega_1}^\bullet$ *satisfies conditions* (I)-(IV);

(ii) $\mathscr{R}_{\omega_1}^\bullet \supset \mathscr{R}$;

(iii) $(\mathscr{R}_{\omega_1}^\bullet)^\bullet = \mathscr{R}_{\omega_1}^\bullet$;

(iv) *if* $\tilde{\mathscr{R}}$ *is a Rényi space such that* $\mathscr{R}\subset\tilde{\mathscr{R}}$ *and* $\tilde{\mathscr{R}}^\bullet = \tilde{\mathscr{R}}$, *then* $\mathscr{R}_{\omega_1}^\bullet \subset \tilde{\mathscr{R}}$.

Proof. To prove (i) note that if $B\in\mathscr{B}_\alpha^\bullet$ or $B, B'\in\mathscr{B}_\alpha^\bullet$, then $B\in(\mathscr{B}_\beta^\bullet)^\bullet$ or $B, B'\in(\mathscr{B}_\beta^\bullet)^\bullet$, respectively, for some $\beta<\alpha$, and thus axioms (I)-(IV) hold in $\mathscr{R}_\alpha^\bullet$ by transfinite induction and Theorem 4.3.

By transfinite induction again, one can prove that $\mathscr{R}\subset\mathscr{R}_\alpha^\bullet$ for every ordinal $\alpha\leqslant\omega_1$, and hence, in particular, (ii) and (iv) hold.

Now, suppose that $\{B_i\}$ is a decreasing sequence in $\mathscr{B}_{\omega_1}^\bullet$ such that

$$\prod_{i=1}^{\infty} P_\alpha^\bullet(B_{i+1}\,|\,B_i) > 0.$$

Of course, $B_i\in(\mathscr{B}_{\beta_i}^\bullet)^\bullet$ for $\beta_i<\omega_1$ $(i\in N)$, so there is $\alpha<\omega_1$ such that $B_i\in\mathscr{B}_\alpha^\bullet$ $(i\in N)$. Since

$$\prod_{i=1}^{\infty} P_\alpha^\bullet(B_{i+1}\,|\,B_i) = \prod_{i=1}^{\infty} P_{\omega_1}^\bullet(B_{i+1}\,|\,B_i) > 0,$$

we have

$$\bigcap_{i=1}^{\infty} B_i\in(\mathscr{B}_\alpha^\bullet)^\bullet \subset \mathscr{B}_{\omega_1}^\bullet.$$

Thus we have proved the inclusion $(\mathscr{B}_{\omega_1}^\bullet)^\bullet\subset\mathscr{B}_{\omega_1}^\bullet$. From (4.5) it follows now that $(\mathscr{B}_{\omega_1}^\bullet)^\bullet = \mathscr{B}_{\omega_1}^\bullet$ and $(P_{\omega_1}^\bullet)^\bullet = P_{\omega_1}^\bullet$ on $\mathscr{A}\times(\mathscr{B}_{\omega_1}^\bullet)^\bullet = \mathscr{A}\times\mathscr{B}_{\omega_1}^\bullet$, so (iii) is shown and the proof is completed.

5. Unions. Now, we are going to extend a given Rényi space $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ joining to $\mathscr{B}$ unions $B_* = \bigcup_{i=1}^{\infty} B_i$ of increasing sequences

$\{B_i\}$, $B_i \in \mathscr{B}$, and defining a conditional probability for this extension by (2.5).

By Remark 2.1, we assume that increasing sequences $\{B_i\}$, $B_i \in \mathscr{B}$, satisfy the condition

$$\prod_{i=1}^{\infty} P(B_i \mid B_{i+1}) > 0. \tag{5.1}$$

THEOREM 5.1. *Let $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ be a Rényi space and let*

$$B_* = \bigcup_{i=1}^{\infty} B_i \tag{5.2}$$

for a fixed increasing sequence $\{B_i\}$, $B_i \in \mathscr{B}$, satisfying (5.1). *Suppose that*

(γ) *for each $B \in \mathscr{B}$, $B \subset B_*$, and for each $i \in N$ identity* (1.10) *holds with A^1, $A^2 \subset B \cap B_i$ and $B^1 = B$, $B^2 = B_i$.*

Then the system $\mathscr{R}_* = [\Omega, \mathscr{A}, \mathscr{B}_*, P_*]$, where $\mathscr{B}_* = \mathscr{B} \cup \{B_*\}$, $P_*(A \mid B) = P(A \mid B)$ for $A \in \mathscr{A}$, $B \in \mathscr{B}$, and

$$P_*(A \mid B_*) = \lim_{i \to \infty} P(A \mid B_i) \quad (A \in \mathscr{A}), \tag{5.3}$$

is a Rényi space. Moreover, definition (5.3) *is consistent, i.e., it does not depend on the choice of an increasing sequence $\{B_i\}$, $B_i \in \mathscr{B}$, satisfying* (5.1) *and* (5.2).

Proof (cf. the proof of Theorem 11 in [3]). Given a set $A \in \mathscr{A}$, we define $A^{(1)} = A \cap B_1$ and $A^{(k)} = A \cap B_k \cap (\Omega \setminus B_{k-1})$ for $k > 1$.

Fix $k \in N$. Since $A^{(k)} \subset B_i$ for $i \geqslant k$, the sequence $\{P(A^{(k)} \mid B_i)\}$ $(i = k, k+1, \ldots)$ is non-increasing, and thus the limit

$$\lim_{i \to \infty} P(A^{(k)} \mid B_i) = P_*(A^{(k)} \mid B_*)$$

exists.

Now, for arbitrary $A \in \mathscr{A}$ let

$$\tilde{P}(A \mid B_*) = \sum_{k=1}^{\infty} P_*(A^{(k)} \mid B_*).$$

We are going to show that

$$P_*(A \mid B_*) = \tilde{P}(A \mid B_*) \tag{5.4}$$

for every $A \in \mathscr{A}$.

Fix $i \in N$. By (III_2), we have

$$P(A \mid B_{k+1}) = P(A \mid B_i) \prod_{j=i}^{\infty} P(B_j \mid B_{j+1})$$

for $A \in \mathscr{A}$, $A \subset B_i$, and $k = i, i+1, \ldots$

Hence

$$P_*(A \mid B_*) = P(A \mid B_i) \prod_{j=i}^{\infty} P(B_j \mid B_{j+1}) \tag{5.5}$$

for every $A \in \mathscr{A}$, $A \subset B_i$.

Now, let $A \in \mathscr{A}$ be arbitrary. Since the sets $A^{(n)}$ $(n \in N)$ are disjoint and $\bigcup_{n=1}^{\infty} A^{(n)} = A \cap B_*$, we get

$$P(A \mid B_i) = P\Big(\bigcup_{n=1}^{\infty} A^{(n)} \cap B_i \mid B_i\Big) = \sum_{n=1}^{i} P(A^{(n)} \mid B_i)$$

using (III_1), (II), and (1.4). Hence, by (5.5), we obtain

$$P(A \mid B_i) = \Big[\prod_{j=i}^{\infty} P(B_j \mid B_{j+1})\Big]^{-1} \sum_{n=1}^{i} P_*(A^{(n)} \mid B_*),$$

which implies that the limit $\lim_{i \to \infty} P(A \mid B_i)$ exists and (5.4) holds.

We shall prove that $\mathscr{R}_*$ satisfies axioms (I)-(III).

Axiom (I) is evident because

$$P_*(B_* \mid B_*) = \lim_{i \to \infty} P(B_* \mid B_i) = 1.$$

To prove (II) suppose that $A_n \in \mathscr{A}$ $(n \in N)$, $A_n \cap A_m = \emptyset$ $(n \neq m)$, and let $A = \bigcup_{n=1}^{\infty} A_n$. By (5.5) and the σ-additivity of P, we have

$$P_*(A^{(k)} \mid B_*) = \sum_{n=1}^{\infty} P_*(A_n^{(k)} \mid B_*).$$

Hence, by (5.4),

$$P_*(A \mid B_*) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} P_*(A_n^{(k)} \mid B_*) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} P_*(A_n^{(k)} \mid B_*) = \sum_{n=1}^{\infty} P_*(A_n \mid B_*),$$

as desired.

The proof of (III) will consist of the following three cases: 1° $B = B' = B_*$, 2° $B = B_*$, $B' = \mathscr{B}$, and 3° $B \in \mathscr{B}$, $B' = B_*$.

Case 1° reduces to the equation

$$P_*(A \mid B_*) = P_*(A \cap B_* \mid B_*),$$

which follows, by letting $i \to \infty$, from the equality

$$P(A \mid B_i) = P(A \cap B_* \mid B_i).$$

Similarly, case 2° can be derived from the identity

$$P(A \mid B_i) = \frac{P(A \cap B_i \mid B')}{P(B_i \mid B')}$$

which holds for sufficiently large n because

$$P_*(B_* \mid B') = \lim_{i\to\infty} P(B_i \mid B') > 0.$$

In case 3°, note that (γ) yields, for fixed $n \in N$, the identity

$$P(A \cap B \mid B)\cdot P(B_n \cap B \mid B_i) = P(B_n \cap B \mid B)\cdot P(A \cap B \mid B_i),$$

where $A \in \mathscr{A}$, $A \cap B \subset B_i$, and $i \geqslant n$.

Letting $i \to \infty$, we get the equation

$$P(A \mid B)\cdot P_*(B_n \cap B \mid B_*) = P(B_n \mid B)\cdot P_*(A \cap B \mid B_*) \tag{5.6}$$

or, using the notation

$$c_1 = P_*(B_n \cap B \mid B_*), \quad c_2 = P(B_n \mid B),$$
$$\mu_1(A) = P(A \mid B), \quad \mu_2(A) = P_*(A \mid B_*)$$

(μ_1, μ_2 are measures on $\mathscr{A}$ by (III)), the equation

$$c_1 \mu_1(A) = c_2 \mu_2(A), \tag{5.7}$$

where (5.6) and (5.7) hold on the ring $\mathscr{A}_1$ of all sets $A \in \mathscr{A}$ such that $A \cap B \subset B_i$ for some $i \in N$, and thus hold for all $A \in \mathscr{A}$ because $\mathscr{A}$ is the smallest σ-ring containing $\mathscr{A}_1$; in fact,

$$A = [A \cap (\Omega \backslash B_*)] \cup [\bigcup_{i=1}^{\infty} (A \cap B_i)]$$

and $A \cap (\Omega\backslash B_*) \in \mathscr{A}_1$, $A \cap B_i \in \mathscr{A}_1$ ($i \in N$) for every $A \in \mathscr{A}$.

Passing to the limits in (5.6) as $n \to \infty$, we get

$$P(A \mid B)\cdot P_*(B \mid B_*) = P_*(A \cap B \mid B_*) \quad (A \in \mathscr{A})$$

using (I) and the inclusion $B \subset B_*$. The proof of (III) is complete.

Finally, suppose that $\{\bar{B}_i\}$ is another increasing sequence in $\mathscr{B}$ satisfying (5.1) and (5.2). By the first part of the proof, the limits

$$\nu_1(A) = \lim_{i\to\infty} P(A \cap B_* \mid B_i), \quad \nu_2(A) = \lim_{i\to\infty} P(A \cap B_* \mid \bar{B}_i)$$

exist for each $A \in \mathscr{A}$ and ν_1, ν_2 are measures on $\mathscr{A}$. They coincide on the ring $\mathscr{A}_2$ of all $A \in \mathscr{A}$ such that $A \cap B_* \subset B_n \cap \bar{B}_n$ for some $n \in N$. In fact, by (γ), we have

$$P(A \cap B_* \mid B_i)\cdot P(B_i \cap \bar{B}_i \mid \bar{B}_i) = P(B_i \cap \bar{B}_i \mid B_i)\cdot P(A \cap B_* \mid B_i),$$

provided $A \cap B_* \subset B_i \cap \bar{B}_i$ and, by Theorem 2.1, $\nu_1 = \nu_2$ on $\mathscr{A}_2$.

As previously, we conclude that $\nu_1 = \nu_2$ on $\mathscr{A}$. It remains to note that

$$\nu_1(A) = \lim_{i\to\infty} P(A \mid B_i) \quad \text{and} \quad \nu_2(A) = \lim_{i\to\infty} P(A \mid \bar{B}_i).$$

Corollary 5.1 (see [3], Theorem 11). *Let $\mathscr{R}$ be a Rényi space with a fixed increasing sequence $\{B_i\}$, $B_i \in \mathscr{B}$, satisfying* (5.1) *and let B_* be of the form* (5.2). *Suppose that*

(γ') *$B \cap B_i \in \mathscr{B}$ provided $B \in \mathscr{B}$ and $P(B \mid B_i) > 0$ for $i \in N$.*

Then $\mathscr{R}_*$ is a Rényi space and definition (5.3) *is consistent.*

Proof. By Theorem 1.4, condition (γ') implies (γ) and the assertion follows by Theorem 5.1.

Remark 5.1. In the case $B_* \in \mathscr{B}$, the statements of Theorem 5.1 and Corollary 5.1 are obvious by Theorem 2.1.

Remark 5.2. The statement of Corollary 5.1 is formulated in [3] (p. 298) as Theorem 11. However, the proof given there is not correct because, when checking case 3° of axiom (III) (i.e., for $B \in \mathscr{B}$ and $B' = B_*$), the first part of Theorem 1.4 is used, although its assumptions are not satisfied. Namely, from the assumptions $A \subset B_N \cap B$ and $P(B \mid B_N) > 0$ the formula

$$\frac{P(A \mid B_N)}{P(B \mid B_N)} = \frac{P(A \mid B)}{P(B \mid B)}$$

is deduced (see [3], p. 300, lines 10-5 from below). But this is true only under the additional assumption that $B \subset B_N$, which is not satisfied in general.

Moreover, there is no proof of the consistency of definition (5.3) in [3].

These two gaps are completed in the second part of the proof of Theorem 5.1. Its first part is a slight modification of the proof given in [3] (p. 298-300).

Now, given a Rényi space $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$, fix an arbitrary family $\mathscr{B}_*$ of sets B_* of the form (5.2), where $\{B_i\}$, $B_i \in \mathscr{B}$, is an increasing sequence satisfying (5.1), and define $P_*(A \mid B_*)$ for all $A \in \mathscr{A}$ and $B_* \in \mathscr{B}_*$ by (5.3). For such a fixed system, we shall apply in the sequel the notation $\mathscr{R}_* = [\Omega, \mathscr{A}, \mathscr{B}_*, P_*]$.

In particular, if $\mathscr{B}_*$ consists of all sets B_* as above, we denote the respective system by $\mathscr{R}^* = [\Omega, \mathscr{A}, \mathscr{B}^*, P^*]$.

Now, we denote by (γ_*) and (γ^*) the analogues of condition (γ) formulated for all B_* belonging to $\mathscr{B}_*$ and $\mathscr{B}^*$, respectively (cf. conditions (β_*) and (β^*) in Section 4).

Theorem 5.2. *Let $\mathscr{R}$ be a Rényi space. The system $\mathscr{R}_*$ is a Rényi space iff $\mathscr{R}$ satisfies condition* (γ_*). *In particular, $\mathscr{R}^*$ is a Rényi space iff $\mathscr{R}$ satisfies* (γ^*). *Moreover,*

$$\mathscr{R} \subset \mathscr{R}_* \subset \mathscr{R}^*. \tag{5.8}$$

Proof. In a similar way as in the proof of Theorem 4.2 (see also Theorem 3.2), we can deduce from Theorem 5.1 that if $\mathscr{R}$ satisfies (γ_*), then $\mathscr{R}_*$ is a Rényi space.

Assume that $\mathscr{R}_*$ is a Rényi space. Let $B_* \in \mathscr{B}_*$ and let $\{B_i\}$, $B_i \in \mathscr{B}$, be an increasing sequence satisfying (5.1) and (5.2). Further, let $B \in \mathscr{B}$, $B \subset B_*$, $A^1, A^2 \in \mathscr{A}$, and $A^1, A^2 \subset B \cap B_n$ for fixed $n \in N$.

By (5.1), we have $P_*(B_i \mid B_*) > 0$ for $i \in N$ (cf. Remark 2.1).

If $P(B \mid B_i) = 0$ for all $i \in N$, then $P(A^j \mid B_n) = 0$ for $j = 1, 2$ by (1.5) and (III_1). Therefore (1.10) holds with $B^1 = B$ and $B^2 = B_n$.

Suppose now that $P(B \mid B_{i_0}) > 0$ for some $i_0 \in N$. Then

$$P_*(B \mid B_*) \geqslant P_*(B \cap B_{i_0} \mid B_*) = P(B \cap B_{i_0} \mid B_{i_0}) \cdot P_*(B_{i_0} \mid B_*) > 0$$

by (1.5), (III_2), and (III_1).

Using (III), we have

$$P(A^j \mid B) \cdot P_*(B \mid B_*) = P_*(A^j \mid B_*), \qquad P(A^j \mid B_n) \cdot P_*(B_n \mid B_*) = P_*(A^j \mid B_*)$$

for $j = 1, 2$, and hence

$$P(A^1 \mid B) \cdot P(A^2 \mid B_n) = \frac{P_*(A^1 \mid B_*) \cdot P_*(A^2 \mid B_*)}{P_*(B \mid B_*) \cdot P_*(B_n \mid B_*)} = P(A^2 \mid B) \cdot P(A^1 \mid B_n),$$

i.e., condition (γ_*) holds in $\mathscr{R}$.

Thus the first equivalence is proved. The second equivalence is a consequence of the first one.

Since $\mathscr{B} \subset \mathscr{B}_*$ and, by Corollary 2.1, $P = P_*$ on $\mathscr{A} \times \mathscr{B}$, we have $\mathscr{R} \subset \mathscr{R}_*$. The inclusion $\mathscr{R}_* \subset \mathscr{R}^*$ is obvious, so (5.8) holds and the theorem is proved.

Remark 5.3. A similar asymmetry as that noted in Remark 4.2 appears for the extensions $\mathscr{R}_*$ and $\mathscr{R}^*$. Namely, to prove that $\mathscr{R}_*$ (or, in particular, $\mathscr{R}^*$) is a Rényi space it suffices to assume (γ_*) (or (γ^*), respectively) only for fixed representations of sets B_* by increasing sequences $\{B_i\}$ satisfying (5.1), (5.2) and for almost all $i \in N$. On the other hand, if $\mathscr{R}_*$ (in particular, $\mathscr{R}^*$) is a Rényi space, then (γ_*) (or (γ^*), respectively) holds for all representations of B_* by increasing sequences $\{B_i\}$ satisfying (5.1), (5.2) and for all $i \in N$.

THEOREM 5.3. *If a Rényi space $\mathscr{R}$ satisfies axiom* (IV), *then $\mathscr{R}^*$ is also a Rényi space satisfying* (IV) *and* (5.8) *holds.*

Proof. Since (IV) implies (γ^*), $\mathscr{R}^*$ is a Rényi space by Theorem 5.1, so it suffices to show that

$$P^*(A^1 \mid B_*) \cdot P(A^2 \mid B) = P^*(A^2 \mid B_*) \cdot P(A^1 \mid B) \tag{5.9}$$

for every $B \in \mathscr{B}$, $B_* \in \mathscr{B}^*$, and $A^1, A^2 \in \mathscr{A}$ such that $A^1, A^2 \subset B \cap B_*$. By (IV) and the definition of P^*, (5.9) holds for all $A^1, A^2 \in \mathscr{A}$ such that $A^1 \subset B \cap B_i$ and $A^2 \subset B \cap B_k$ for some $i, k \in N$. Hence the general case follows by applying analogous considerations as those at the end of the proof of Theorem 5.1.

THEOREM 5.4. *If a Rényi space $\mathcal{R}$ satisfies* (IV′), *then $\mathcal{R}^*$ is a Rényi space satisfying* (IV′) *and* (5.8) *holds.*

Proof. By Theorem 5.3, it suffices to prove (IV′) for $\mathcal{R}^*$.

Suppose that $B_*^1, B_*^2 \in \mathcal{B}^*$ and

$$P^*(B_*^1 \mid B_*^2) + P^*(B_*^2 \mid B_*^1) > 0.$$

We can assume that $P^*(B_*^1 \mid B_*^2) > 0$. We have

$$B_*^j = \bigcup_{i=1}^{\infty} B_i^j, \quad B_i^j \in \mathcal{B}, \quad B_i^j \subset B_{i+1}^j$$

for $i \in N$, $j = 1, 2$, and

$$\prod_{i=1}^{\infty} P(B_i^j \mid B_{i+1}^j) > 0 \quad (j = 1, 2).$$

Moreover,

$$\lim_{i \to \infty} P(B_i^1 \mid B_i^2) = P^*(B_*^1 \mid B_*^2)$$

by Theorem 2.1. Consequently, $P(B_i^1 \mid B_i^2) > 0$ and $\bar{B}_i = B_i^1 \cap B_i^2 \in \mathcal{B}$ for sufficiently large i, since $\mathcal{R}$ fulfils (IV′).

As $\bar{B}_i \subset \bar{B}_{i+1}$ ($i \in N$) and $\bigcup_{i=1}^{\infty} B_i = B_*^1 \cap B_*^2$, it remains to prove that

$$\prod_{i=k}^{\infty} P(\bar{B}_i \mid \bar{B}_{i+1}) > 0 \tag{5.10}$$

for sufficiently large $k \in N$.

But, using (III), we have

$$\prod_{i=k}^{n} P(\bar{B}_i \mid \bar{B}_{i+1}) = \prod_{i=k}^{n} \frac{P^*(\bar{B}_i \mid B_*^2)}{P^*(\bar{B}_{i+1} \mid B_*^2)} = \frac{P^*(\bar{B}_k \mid B_*^2)}{P^*(\bar{B}_{n+1} \mid B_*^2)}$$

for $n > k$, so letting $n \to \infty$ we get (5.10) for sufficiently large k because

$$\lim_{j \to \infty} P^*(\bar{B}_j \mid B_*^2) = P^*(B_*^1 \mid B_*^2) > 0.$$

The proof is complete.

THEOREM 5.5. *Suppose that a Rényi space $\mathcal{R} = [\Omega, \mathcal{A}, \mathcal{B}, P]$ satisfies* (IV) (*or* (IV′)) *and is additive, i.e.,*

(+) *$B^1, B^2 \in \mathcal{B}$ implies $B^1 \cup B^2 \in \mathcal{B}$.*

Then $\mathcal{R}^*$ is a Rényi space satisfying (IV) (*or* (IV′), *respectively*) *and*

$$\mathcal{R}^{**} = \mathcal{R}^*. \tag{5.11}$$

Proof. By Theorems 5.3 and 5.4, it suffices to prove (5.11).

Let $B_{**} \in \mathscr{B}^{**}$. This means that $B_{**} = \bigcup_{i=1}^{\infty} B^i_*$, where $B^i_* \in \mathscr{B}^*$, $B^i_* \subset B^{i+1}_*$ for $i \in N$ and

$$\prod_{i=1}^{\infty} P^*(B^i_* \mid B^{i+1}_*) > 0. \tag{5.12}$$

Further, we have $B^i_* = \bigcup_{j=1}^{\infty} B^i_j$ for $i \in N$, where

$$B^i_j \in \mathscr{B},\ B^i_j \subset B^i_{j+1} \text{ for } i, j \in N, \quad \prod_{k=1}^{\infty} P(B^i_k \mid B^i_{k+1}) > 0 \text{ for } i \in N. \tag{5.13}$$

Since

$$\lim_{j \to \infty} P^*(B^i_j \mid B^i_*) = 1 \quad \text{for } i \in N,$$

we can assume that

$$P^*(B^i_i \mid B^i_*) > 1 - 1/i^2 \quad \text{for } i \in N. \tag{5.14}$$

Putting $\bar{B}_k = \bigcup_{i=1}^{k} B^i_k$, we have

$$\bar{B}_k \in \mathscr{B},\ \bar{B}_k \subset \bar{B}_{k+1} \text{ for } k \in N \quad \text{and} \quad \bigcup_{j=1}^{\infty} \bar{B}_j = B_{**} \tag{5.15}$$

by (5.13) and (+).

We shall show that

$$\prod_{k=1}^{\infty} P(\bar{B}_k \mid \bar{B}_{k+1}) > 0. \tag{5.16}$$

Since $B^k_k \subset B^k_* \subset B^{k+1}_*$ and $B^k_k \subset \bar{B}_k \subset \bar{B}_{k+1} \subset B^{k+1}_*$, we get

$$P^*(B^k_k \mid B^k_*) \cdot P^*(B^k_* \mid B^{k+1}_*) = P^*(B^k_k \mid B^{k+1}_*) \leqslant P(\bar{B}_k \mid \bar{B}_{k+1})$$

in view of (III_2), (1.5), and (1.6).

Hence

$$\prod_{k=1}^{\infty} P(\bar{B}_k \mid \bar{B}_{k+1}) \geqslant \prod_{k=1}^{\infty} P^*(B^k_k \mid B^k_*) \cdot \prod_{k=1}^{\infty} P^*(B^k_* \mid B^{k+1}_*) > 0$$

by (5.12) and (5.14), and thus (5.16) is shown.

This and (5.15) yield $B_{**} \in \mathscr{B}^*$ and, consequently, the inclusion $\mathscr{B}^{**} \subset \mathscr{B}^*$ is proved. The converse inclusion and the identity $P^* = P^{**}$ follow now, by (5.8), and the proof is complete.

COROLLARY 5.2. *Let* $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ *be a Rényi space satisfying* (IV) *or* (IV′) *and the following condition of quasi-additivity*:

($\dot{+}$) *For any* $B^1, B^2 \in \mathscr{B}$ *there exists* $B \in \mathscr{B}$ *such that* $B^1 \cup B^2 \subset B$ *and* $P(B^1 \cup B^2 \mid B) > 0$.

Then $\mathscr{R}^{\circ **} = \mathscr{R}^{\circ *}$. *If, additionally,* $\mathscr{R}^{\circ} = \mathscr{R}$, *then* (5.11) *holds.*

Proof. It suffices to notice that the Rényi space $\mathscr{R}^{\circ}$ fulfils the assumptions of Theorem 5.5.

Remark 5.4. Note that condition $(\dot{+})$ can be written in the following equivalent form:

$(\dot{+})$ *For any* B^1, $B^2 \in \mathscr{B}$ *there is* $B \in \mathscr{B}$ *such that* $B^1 \cup B^2 \subset B$ *and* $P(B^1 \mid B) + P(B^2 \mid B) > 0$.

Conditions $(+)$ and $(\dot{+})$ are considered, though in another context, in [1] (p. 356).

The following example shows that conditions (γ), (γ'), (γ_*), (γ^*), (IV) and (IV′) cannot be omitted in Theorem 5.1, Corollary 5.1 and Theorems 5.2, 5.3, 5.4, respectively, and that condition (IV) cannot be replaced by $\mathrm{IV}_1^{(2)}$ in Theorem 5.3.

Example 5.1. Let $\Omega = [0, 1]$ and let $\mathscr{A}$ be the family of all Borel subsets of Ω. Putting $x_i = 1 - 2^{-i}$ for $i = 0, 1, 2, \ldots$, we define

$$B = \{x_i \colon i \in N\}, \quad B_n = \{x_i \colon i = 0, 1, 2, \ldots, n\}, \quad \mathscr{B} = \{B, B_n \colon n \in N\},$$

$$B_* = \bigcup_{n=1}^{\infty} B_n = \{x_k \colon k = 0, 1, 2, \ldots\}, \quad P(A \mid B) = \delta_{x_1}(A),$$

and

$$P(A \mid B_n) = (1 - 2^{-n})^{-1} \sum_{i=1}^{n} 2^{-i} \delta_{x_i}(A)$$

for $A \in \mathscr{A}$, where δ_c is the probability measure concentrated at c.

It is clear that $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ is a Rényi space and the sequence $\{B_i\}$ fulfils (5.1). Equation (1.10) with $A^1 = \{x_1\}$, $A^2 = \{x_2\}$ and $B^1 = B$, $B^2 = B_i$ does not hold for any $i = 2, 3, \ldots$ This means that conditions (γ), (γ_*), (γ^*) and (IV) are not satisfied because $B \subset B_*$ and $A^1, A^2 \subset B \cap B_i$ for $i \geqslant 2$. It is easy to see that conditions (γ') and (IV′) do not hold either, but condition $\mathrm{IV}_1^{(2)}$ does.

Putting $\mathscr{B}_* = \mathscr{B}^* = \mathscr{B} \cup \{B_*\}$, $P_* = P^* = P$ on $\mathscr{A} \times \mathscr{B}$, and

$$P_*(A \mid B_*) = P^*(A \mid B_*) = \lim_{i \to \infty} P(A \mid B_i) = \sum_{i=1}^{\infty} 2^{-i} \delta_{x_i}(A),$$

we have $B \subset B_*$ and $P_*(B \mid B_*) = P^*(B \mid B_*) = 1 > 0$, but

$$P_*(A \mid B) = P^*(A \mid B) \neq \frac{P_*(A \cap B \mid B_*)}{P_*(B \mid B_*)} = \frac{P^*(A \cap B \mid B_*)}{P^*(B \mid B_*)}$$

if $A = \{x_2\}$, for instance. Thus the system $\mathscr{R}_* = \mathscr{R}^*$ is not a Rényi space.

Now, replacing in Example 4.2 the sets B_n by the sets $[0, 1/2] \times \times [0, 1 - 1/n]$, we get an example which shows that condition (γ^*) is essentially weaker than (IV).

The following example proves that axiom (IV′) is not sufficient for relation (5.11) to hold.

Example 5.2. Let $\Omega = [0, 1]^2$ and let $\mathscr{A}$ be the family of all Borel subsets of Ω. Now, let

$$B_{ij} = \left(\left[0,\, 1-\frac{1}{2i}\right]\times\left[0,\, 1-\frac{1}{2j}\right]\right)\cup\left(\left[1-\frac{1}{2i},\, 1\right]\times[0,\, c_{ij}]\right)$$

for $i, j \in N$, where

$$c_{ij} = \frac{1}{2}\left(1-\frac{1}{i}+\frac{1}{i(i+1)j}\right),$$

and let

$$\mathscr{B} = \{B_{ij}\colon\ i, j \in N\} \quad\text{and}\quad P(A\,|\,B_{ij}) = \frac{|A \cap B_{ij}|}{|B_{ij}|}$$

for $i, j \in N$ and $A \in \mathscr{A}$.

One can check that $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ fulfils axioms (I)-(III). Moreover, since $i < i'$ implies $c_{ij} < c_{i'j'}$ for any $j, j' \in N$, we have $B_{ij} \cap B_{i'j'} \in \mathscr{B}$ for arbitrary $i, j, i', j' \in N$, i.e., (IV′) holds.

We have

$$\mathscr{B}^* = \mathscr{B} \cup \{B_j\colon\ j \in N\} \quad\text{and}\quad \mathscr{B}^{**} = \mathscr{B}^* \cup \{B\},$$

where

$$B_j = [0, 1]\times[0, 1-1/2j] \quad\text{and}\quad B = [0, 1]^2.$$

Consequently, $\mathscr{B}^* \neq \mathscr{B}^{**}$.

Now, suppose that a fixed Rényi space $\mathscr{R} = [\Omega, \mathscr{A}, \mathscr{B}, P]$ satisfies axiom (IV) and define, by transfinite induction, the α-th iteration of the operation $*$ for any ordinal α: $\mathscr{R}_0^* = \mathscr{R}$; if $\mathscr{R}_\beta^* = [\Omega, \mathscr{A}, \mathscr{B}_\beta^*, P_\beta^*]$ for all $\beta < \alpha$ satisfy (I)-(IV) and

$$\mathscr{R}_\beta^* \subset \mathscr{R}_{\beta'}^* \quad \text{for } \beta < \beta' < \alpha, \tag{5.17}$$

then we put $\mathscr{R}_\alpha^* = \bigcup\limits_{\beta<\alpha} (\mathscr{R}_\beta^*)^*$, i.e., $\mathscr{B}_\alpha^* = \bigcup\limits_{\beta<\alpha} (\mathscr{B}_\beta^*)^*$ and

$$P_\alpha^*(A\,|\,B) = P_\beta^*(A\,|\,B) \quad \text{for } A \in \mathscr{A},\ B \in (\mathscr{B}_\beta^*)^*. \tag{5.18}$$

Definition (5.18) is consistent by (5.17).

As in Theorem 4.5, we can prove the following result:

THEOREM 5.6. *Given a Rényi space $\mathscr{R}$ with property* (IV), *the system $\mathscr{R}_{\omega_1}^*$ is the smallest Rényi space containing $\mathscr{R}$ and closed with respect to the operation $*$. More precisely*:

(i) *$\mathscr{R}_{\omega_1}^*$ satisfies conditions* (I)-(IV);

(ii) $\mathscr{R}_{\omega_1}^* \supset \mathscr{R}$;

(iii) $(\mathscr{R}^*_{\omega_1})^* = \mathscr{R}^*_{\omega_1}$;

(iv) *if* $\tilde{\mathscr{R}}$ *is a Rényi space and* $\mathscr{R} \subset \tilde{\mathscr{R}}$, $\tilde{\mathscr{R}}^* = \tilde{\mathscr{R}}$, *then* $\mathscr{R}^*_{\omega_1} \subset \tilde{\mathscr{R}}$.

Remark 5.5. In Theorems 4.5 and 5.6, property (IV) can be replaced by (IV′).

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Added in proof. The remark at the end of Introduction is proved in [5]. Namely, given a Rényi space satisfying (IV), $(\mathscr{R}^\circ)^*_{\omega_1}$ is the smallest Rényi space fulfilling (IV), containing $\mathscr{R}$ and closed with respect to the operations °, •, and * (Theorem 2 in [5]). On the other hand, $(\mathscr{R}^\circ)^*_\alpha$ does not have these properties for $\alpha < \omega_1$ in general, which is shown in Theorem 4 of [6].

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