COLLOQUIUM MATHEMATICUM

VOL. LV 1988 FASC. 1

ON SOME INTEGRAL INEQUALITIES OF BLOCK TYPE

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In recent papers [2] and [3] the method of derivation of the integral inequalities of Hardy type, i.e., integral inequalities of the form

$$(1)\qquad \int_I s|h|^p\,dt \leqslant \int_I r|\dot{h}|^p\,dt,$$

where $p > 1$, $I = (\alpha, \beta)$, $-\infty \leqslant \alpha < \beta \leqslant \infty$, and r, s are some fixed functions, was presented and the classes of functions h for which the inequalities (1) hold were examined. In this paper the same method will be used to obtain and examine certain integral inequalities of Block type, i.e., integral inequalities of the form

$$(2)\qquad v_j(x)|h(x)|^p \leqslant \int_I (r|\dot{h}|^p + s|h|^p)\,dt,$$

where $x \in I$ and r, s, v_j are some fixed functions. The inequalities of the form (2) were considered by Block [1] and Redheffer [4], [5].

Let us denote by $\mathrm{AC}(I)$ the class of all real functions defined and absolutely continuous on the open interval $I = (\alpha, \beta)$, $-\infty \leqslant \alpha < \beta \leqslant \infty$. Let p be any real number such that $p > 1$ and let $r \in \mathrm{AC}(I)$ and $\varphi \in \mathrm{AC}(I)$ be such that $r > 0$, $\varphi > 0$ in I and $r|\dot{\varphi}|^{p-1}\operatorname{sgn}\dot{\varphi} \in \mathrm{AC}(I)$. Let us put

$$(3)\qquad s \equiv (r|\dot{\varphi}|^{p-1}\operatorname{sgn}\dot{\varphi})^{\cdot}\,\varphi^{1-p} \quad \text{and} \quad v \equiv r|\dot{\varphi}|^{p-1}\operatorname{sgn}\dot{\varphi}\cdot\varphi^{1-p}.$$

and denote by $\hat{H}$ the class of functions $h \in \mathrm{AC}(I)$ satisfying the following conditions:

$$(4)\qquad \int_I r|\dot{h}|^p\,dt < \infty, \quad \int_I s|h|^p\,dt < \infty,$$

$$(5)\qquad \liminf_{t\to\alpha+} v|h|^p < \infty, \quad \limsup_{t\to\beta-} v|h|^p > -\infty.$$

Under the above assumptions ([2], Theorem 1 and Lemma 2) the following is true:

THEOREM 1. *For every function* $h \in \hat{H}$ *the following statements are valid*:
(i) *The limits in* (5) *are proper and finite, and*

(6) $$\lim_{t \to \beta -} v|h|^p - \lim_{t \to \alpha +} v|h|^p \leqslant \int_I (r|\dot{h}|^p + s|h|^p)\,dt.$$

(ii) *If* $h \not\equiv 0$, *then* (6) *becomes an equality if and only if* $h = c\varphi$ *with* $c = \text{const} \neq 0$ *and the additional conditions*

$$\int_I r|\dot{\varphi}|^p\,dt < \infty, \qquad \int_I |s|\,\varphi^p\,dt < \infty$$

are satisfied.

Now, enlarging the class of the above functions r and φ, we derive the integral inequalities of the form (2).

Let us fix a point $x \in I$ and let $I_1 = (\alpha, x)$, $I_2 = (x, \beta)$. Now, let r and φ be some given functions such that $r > 0$, $\varphi > 0$ in $I_1 \cup I_2$, $r \in \mathrm{AC}(I_k)$, $\varphi \in \mathrm{AC}(I_k)$, $r|\dot{\varphi}|^{p-1}\operatorname{sgn}\dot{\varphi} \in \mathrm{AC}(I_k)$ for $k = 1, 2$, and such that

(7) $$\limsup_{t \to x-} v > -\infty, \qquad \liminf_{t \to x+} v < \infty,$$

where v is defined by (3).

Let $v_j(x) \equiv \limsup_{t \to x-} v - \liminf_{t \to x+} v$ denote the jump of v at the point $x \in I$. It follows from (7) that $v_j(x) > -\infty$. If $v_j(x) = \infty$, then in the sequel we assume that

$$v_j(x)\,|h(x)|^p = 0 \quad \text{if } h(x) = 0$$

and that

$$v_j(x)\,|h(x)|^p = \infty \quad \text{if } h(x) \neq 0.$$

Further, let $\hat{H}$ denote the class of functions $h \in \mathrm{AC}(I)$ defined as previously, where s is defined by (3).

THEOREM 2. *For every function* $h \in \hat{H}$ *the following statements are valid*:
(i) *The limits in* (5) *are proper and finite, and*

(8) $$v_j(x)\,|h(x)|^p + \lim_{t \to \beta -} v|h|^p - \lim_{t \to \alpha +} v|h|^p \leqslant \int_I (r|\dot{h}|^p + s|h|^p)\,dt.$$

(ii) *If* $h \not\equiv 0$, *then* (8) *becomes an equality if and only if* $h = c\hat{\varphi}$, *where* $c = \text{const} \neq 0$ *and* $\hat{\varphi} \in \mathrm{AC}(I)$ *is such that* $\hat{\varphi} \not\equiv 0$ *in* I *and in each of the intervals* I_k, $k = 1, 2$, *we have either* $\hat{\varphi} = c_k\varphi$, *where* $c_k = \text{const} \neq 0$, *provided that the conditions*

$$\int_{I_k} r|\dot{\varphi}|^p\,dt < \infty, \qquad \int_{I_k} |s|\,\varphi^p\,dt < \infty$$

are satisfied, or $\hat{\varphi} \equiv 0$ *on* I_k.

Proof. Let $h \in \hat{H}$. Then $h \in AC(I)$ and by (7) we have

$$\limsup_{t \to x-} v|h|^p > -\infty, \quad \liminf_{t \to x+} v|h|^p < \infty. \tag{9}$$

Hence, by (5), the assumptions of Theorem 1 are satisfied in each of the intervals $I_1 = (\alpha, x)$ and $I_2 = (x, \beta)$. Thus the limits in (5) and (9) are proper and finite, and (6) holds both in I_1 and I_2. Adding both sides of those inequalities we obtain

$$\lim_{t \to x-} v|h|^p - \lim_{t \to x+} v|h|^p + \lim_{t \to \beta-} v|h|^p - \lim_{t \to \alpha+} v|h|^p \leqslant \int_I (r|\dot{h}|^p + s|h|^p)\,dt. \tag{10}$$

By the continuity of h at the point x and from the existence of finite limits $\lim_{t \to x-} v|h|^p$ and $\lim_{t \to x+} v|h|^p$ it follows that if $\hat{H}$ contains a function which is non-zero at x, then there exist finite limits $\lim_{t \to x-} v$ and $\lim_{t \to x+} v$ and, moreover, $v_j(x) < \infty$. Hence, in that case we have

$$\lim_{t \to x-} v|h|^p - \lim_{t \to x+} v|h|^p = v_j(x)|h(x)|^p. \tag{11}$$

On the other hand, if all functions in $\hat{H}$ are zero at x, then $v_j(x) = \infty$ and

$$\lim_{t \to x-} v|h|^p - \lim_{t \to x+} v|h|^p \geqslant 0 = v_j(x)|h(x)|^p. \tag{12}$$

Indeed, if

$$\lim_{t \to x-} v|h|^p \neq 0,$$

then, in view of (7),

$$\lim_{t \to x-} v = \infty$$

must hold, and therefore

$$\lim_{t \to x-} v|h|^p > 0.$$

Similarly we deduce that

$$\lim_{t \to x+} v|h|^p \leqslant 0.$$

Now (8) follows from (10)–(12), which proves (i).

If for some function $h \in \hat{H}$ we have an equality in (8), then it follows from (11) and (12) that an equality in (10) holds; since (6) is valid in every interval I_k, $k = 1, 2$, we must have an equality in (6). Using Theorem 1 (ii) we infer that (6) becomes an equality for $h \not\equiv 0$ if and only if $h = c_k \varphi$ in I_k, where $c_k = \text{const} \neq 0$ and φ satisfies the additional conditions

$$\int_{I_k} r|\dot{\varphi}|^p dt < \infty, \qquad \int_{I_k} |s| \varphi^p dt < \infty.$$

Hence (ii) follows.

Let us denote by $\tilde{H}$ the class of functions $h \in \hat{H}$ which satisfy the additional limit condition

$$\liminf_{t \to \alpha+} v|h|^p \leqslant \limsup_{t \to \beta-} v|h|^p. \tag{13}$$

The condition (13) is equivalent to

$$\lim_{t \to \alpha+} v|h|^p \leqslant \lim_{t \to \beta-} v|h|^p. \tag{13'}$$

THEOREM 3. *For every function* $h \in \tilde{H}$ *the inequality*

$$v_J(x)|h(x)|^p \leqslant \int_I (r|\dot{h}|^p + s|h|^p)\, dt \tag{14}$$

holds. The inequality (14) *becomes an equality if and only if* $h = c\hat{\varphi}$, *where* $c = \text{const}$ *and* $\hat{\varphi}$ *is a function satisfying all the conditions of Theorem* 2 (ii) *and the following additional condition*:

$$\lim_{t \to \alpha+} v|\hat{\varphi}|^p = \lim_{t \to \beta-} v|\hat{\varphi}|^p.$$

Proof. By virtue of (13′) and Theorem 2 (i), the inequality (14) follows from (8). If both sides of (14) are equal for some non-vanishing function $h \in \tilde{H}$, then by (8) and (13′) we have

$$\lim_{t \to \alpha+} v|h|^p = \lim_{t \to \beta-} v|h|^p.$$

Applying Theorem 2 (ii) we get $h = c\hat{\varphi}$, where $c = \text{const} \neq 0$ and $\hat{\varphi}$ is a function satisfying all the conditions of Theorem 2 (ii). The theorem follows now easily.

Inequalities of the form (14) are called the *inequalities of Block type* (see [5]). They are some generalizations of the inequalities considered by Block in [1].

Now, we describe the class $\tilde{H}$ in the most frequently occurring cases. Let $q = p/(p-1) > 1$ denote the conjugate of $p > 1$. We assume in the sequel that $s \geqslant 0$ almost everywhere in $I = (\alpha, \beta)$ and that the proper limits

$$\lim_{t \to \alpha+} v \equiv v(\alpha) \quad \text{and} \quad \lim_{t \to \beta-} v \equiv v(\beta)$$

exist. We introduce the following terminology:

α (resp. β) is of the *I type* if $v(\alpha) \leqslant 0$ (resp. $v(\beta) \geqslant 0$);

α (resp. β) is of the *II type* if $0 < v(\alpha) < \infty$ (resp. $-\infty < v(\beta) < 0$) and

$$\int_\alpha^t r^{-q/p}\,dt < \infty \quad (\text{resp. } \int_t^\beta r^{-q/p}\,dt < \infty) \quad \text{for some } t\in I;$$

α (resp. β) is of the *III type* if $v(\alpha) = \infty$ (resp. $v(\beta) = -\infty$) and

$$v(t)\Big(\int_\alpha^t r^{-q/p}\,dt\Big)^{p/q} = O(1) \quad \text{for } t \to \alpha+$$

$$(\text{resp. } v(t)\Big(\int_t^\beta r^{-q/p}\,dt\Big)^{p/q} = O(1) \quad \text{for } t \to \beta-).$$

We denote by H the class of functions $h\in \mathrm{AC}(I)$ satisfying the conditions (4), and by H_0 (resp. H^0) the class of functions $h\in H$ satisfying the condition

$$\liminf_{t\to\alpha+}|h| = 0 \quad (\text{resp. } \liminf_{t\to\beta-}|h| = 0). \tag{15}$$

In the cases considered below, (15) is equivalent to

$$\lim_{t\to\alpha+} h \equiv h(\alpha) = 0 \quad (\text{resp. } \lim_{t\to\beta-} h \equiv h(\beta) = 0). \tag{15'}$$

THEOREM 4. *Let $s \geqslant 0$ almost everywhere in the interval $I = (\alpha, \beta)$.*

(i) *If both points α and β are of the I type, then $\tilde{H} = H$.*

(ii) *If the point α is of the II type and the point β is of the I type, then $\tilde{H} \supset H_0$.*

(iii) *If the point α is of the III type and the point β is of the I type, then $\tilde{H} = H_0$.*

(iv) *If the point α is of the I type and the point β is of the II type, then $\tilde{H} \supset H^0$.*

(v) *If the point α is of the I type and the point β is of the III type, then $\tilde{H} = H^0$.*

(vi) *If both points α and β are of the II or III type, then $\tilde{H} = H_0 \cap H^0$.*

Proof. If $h\in \mathrm{AC}(I)$ and

$$\int_I r|\dot{h}|^p\,dt < \infty \quad \text{and} \quad \int_\alpha^t r^{-q/p}\,dt < \infty \quad \text{for some } t\in I,$$

then by Lemma 3 in [2] it follows that a finite limit value $h(\alpha)$ exists. It is also shown in [2] that if $v(\alpha) > 0$ and $h(\alpha) = 0$, then the estimation

$$0 \leqslant v(t)|h(t)|^p \leqslant v(t)\Big(\int_\alpha^t r^{-q/p}\,dt\Big)^{p/q}\int_\alpha^t r|\dot{h}|^p\,dt \tag{16}$$

is true in some right-hand neighbourhood of the point α.

We show that if $h\in \mathrm{AC}(I)$, $s \geqslant 0$ in I and

$$\int_I s|h|^p\,dt < \infty \quad \text{and} \quad v(\alpha) = 0,$$

then $\liminf_{t\to\alpha+} v|h|^p \leqslant 0$. Indeed, if there exists a sequence $\{t_k\}$ such that $t_k \in I$, $t_k \to \alpha+$ and $v(t_k) \leqslant 0$, then $v(t_k)|h(t_k)|^p \leqslant 0$, and therefore

$$\liminf_{t\to\alpha+} v|h|^p \leqslant 0.$$

Now, let $v > 0$ in some right-hand neighbourhood $U \subset (\alpha, x)$ of the point α. Let us put $w \equiv r|\dot{\varphi}|^{p-1}\operatorname{sgn}\dot{\varphi}$. In that case $s = \dot{w}\varphi^{1-p}$ in I and it follows from the condition $s \geqslant 0$ that $\dot{w} \geqslant 0$ in I. Since, by the assumption, w is absolutely continuous in U, it is non-decreasing in U and the limit $\lim_{t\to\alpha+} w$ exists. On the other hand, we have $v = w\varphi^{1-p}$, and since $v > 0$ in U, we obtain $w > 0$ in U. From the definition of w it follows that $\dot{\varphi} > 0$ in U, and therefore the function φ^{1-p} is decreasing in U and the limit

$$\lim_{t\to\alpha+} \varphi^{1-p} > 0$$

exists because $\varphi > 0$. It follows from the condition

$$v(\alpha) \equiv \lim_{t\to\alpha+} w\varphi^{1-p} = 0$$

that

$$\lim_{t\to\alpha+} w = 0.$$

Now suppose that

$$\liminf_{t\to\alpha+} v|h|^p > 0.$$

Then there are a constant $\sigma > 0$ and a right-hand neighbourhood $U_1 \subset U$ of α such that $v|h|^p \geqslant \sigma$ in U_1. Hence $\varphi^{1-p}|h|^p \geqslant \sigma w^{-1}$ in U_1. Let $a \in U_1$, $t \in U_1$ and $a < t$. We have

$$\int_a^t s|h|^p dt = \int_a^t \dot{w}\varphi^{1-p}|h|^p dt \geqslant \sigma \int_a^t w^{-1}\dot{w} dt = \sigma(\ln w(t) - \ln w(a))$$

and as $a \to \alpha+$ we obtain

$$\int_\alpha^t s|h|^p dt = \infty$$

because $w(a) \to 0+$, which contradicts the condition $\int_I s|h|^p dt < \infty$. Thus

$$\liminf_{t\to\alpha+} v|h|^p \leqslant 0.$$

Now, if α is of the I type and $h \in H$, then $\liminf_{t\to\alpha+} v|h|^p \leqslant 0$, because if $v(\alpha)$

$= 0$, then it is a result of the above considerations, while if $v(\alpha) < 0$, then $v|h|^p \leqslant 0$ in some neighbourhood of α, and hence

$$\liminf_{t \to \alpha+} v|h|^p \leqslant 0.$$

If α is of the II type and $h \in H_0$, then $v|h|^p \geqslant 0$ in some neighbourhood of α, and hence

$$\liminf_{t \to \alpha+} v|h|^p = 0,$$

because $0 < v(\alpha) < \infty$ and $\liminf_{t \to \alpha+} |h| = 0$.

If α is of the II type, β is of the II or III type and $h \in \tilde{H}$, then

$$\lim_{t \to \alpha+} v|h|^p \geqslant 0 \quad \text{and} \quad \lim_{t \to \beta-} v|h|^p \leqslant 0$$

and it follows from (13′) that

$$\lim_{t \to \alpha+} v|h|^p = 0.$$

Since

$$\int_\alpha^t r^{-q/p} dt < \infty \text{ for some } t \in I \quad \text{and} \quad 0 < v(\alpha) < \infty,$$

the finite value $h(\alpha)$ exists and $h(\alpha) = 0$, i.e., $h \in H_0$.

If α is of the III type and $h \in H_0$, then the estimation (16) is true in some neighbourhood of α, and hence

$$\lim_{t \to \alpha+} v|h|^p = 0.$$

If α is of the III type and $h \in \tilde{H}$, then

$$\int_\alpha^t r^{-q/p} dt < \infty \quad \text{for some } t \in I,$$

and hence the finite value $h(\alpha)$ exists. By Theorem 2 (i), a finite limit $\lim_{t \to \alpha+} v|h|^p$ exists for $h \in \tilde{H}$, and hence $h(\alpha) = 0$, because $v(\alpha) = \infty$. Thus $h \in H_0$.

Similar symmetric conclusions are valid if β (α) is substituted for α (β) and the class H^0 for H_0.

Basing on these considerations we can easily derive the theorem.

Now, the applications of Theorems 3 and 4 will be given, and the definite integral inequalities of Block type will be derived by using the method presented above.

EXAMPLE 1. Let $I = (\alpha, \beta)$, $-\infty \leqslant \alpha < \beta \leqslant \infty$, and let $r > 0$ be an arbitrary absolutely continuous function on I such that

$$\int_I r^{-q/p} dt < \infty.$$

Fix $x \in I$. Further, let

$$\varphi = \int_{\alpha}^{t} r^{-q/p}\, dt \text{ in } (\alpha, x) \quad \text{and} \quad \varphi = \int_{t}^{\beta} r^{-q/p}\, dt \text{ in } (x, \beta).$$

In this case $s = 0$ in I and for $\alpha < x < \beta$ we obtain the inequality

$$[(\int_{\alpha}^{x} r^{-q/p}\, dt)^{-p/q} + (\int_{x}^{\beta} r^{-q/p}\, dt)^{-p/q}]\, |h(x)|^p \leqslant \int_{\alpha}^{\beta} r\, |\dot{h}|^p\, dt \tag{17}$$

which is valid for $h \in H_0 \cap H^0$. An equality occurs in (17) only for the function

$$h = c \int_{x}^{\beta} r^{-q/p}\, dt \int_{\alpha}^{t} r^{-q/p}\, dt \quad \text{in } (\alpha, x\rangle$$

and

$$h = c \int_{\alpha}^{x} r^{-q/p}\, dt \int_{t}^{\beta} r^{-q/p}\, dt \quad \text{in } (x, \beta),$$

where $c = \text{const}$.

If we assume that $-\infty < \alpha < \beta < \infty$, $r = 1$ and $p = 2$, then (17) takes the form

$$\frac{\beta - \alpha}{(x-\alpha)(\beta - x)} |h(x)|^2 \leqslant \int_{\alpha}^{\beta} \dot{h}^2\, dt, \quad \alpha < x < \beta, \tag{18}$$

where an equality occurs only for

$$h = c(\beta - x)(t - \alpha) \quad \text{in } (\alpha, x\rangle$$

and

$$h = c(x - \alpha)(\beta - t) \quad \text{in } (x, \beta),$$

where $c = \text{const}$ (see [1]).

EXAMPLE 2. Let $I = (\alpha, \beta)$, $-\infty < \alpha < \beta < \infty$, $r = 1$ in I, $p = 2$ and let $x \in I$ be an arbitrary fixed point and $\lambda > 0$ an arbitrary fixed constant.

Let us put $\varphi = \cosh \lambda(t-\alpha)$ in (α, x) and $\varphi = \cosh \lambda(\beta - t)$ in (x, β). We obtain the inequality

$$\frac{\lambda \sinh \lambda(\beta - \alpha)}{\cosh \lambda(x-\alpha) \cosh \lambda(\beta - x)} |h(x)|^2 \leqslant \int_{\alpha}^{\beta} (\dot{h}^2 + \lambda^2 h^2)\, dt \tag{19}$$

which holds for $\alpha < x < \beta$ and $h \in H$. An equality occurs in (19) only for the function

$$h = c \cosh \lambda(\beta - x) \cosh \lambda(t - \alpha) \quad \text{in } (\alpha, x\rangle$$

and

$$h = c\cosh\lambda(x-\alpha)\cosh\lambda(\beta-t) \quad \text{in } (x, \beta),$$

where $c = \text{const.}$

Let $\varphi = \lambda^{-1}\sinh\lambda(t-\alpha)$ in (α, x) and $\varphi = \cosh\lambda(\beta-t)$ in (x, β). Then we have the inequality

$$(20) \qquad \frac{\lambda\cosh\lambda(\beta-\alpha)}{\sinh\lambda(x-\alpha)\cosh\lambda(\beta-x)}|h(x)|^2 \leqslant \int_\alpha^\beta (\dot{h}^2+\lambda^2 h^2)\,dt$$

which holds for $\alpha < x < \beta$ and $h \in H_0$. An equality occurs in (20) only for the function

$$h = \frac{c}{\lambda}\cosh\lambda(\beta-x)\sinh\lambda(t-\alpha) \quad \text{in } (\alpha, x\rangle$$

and

$$h = \frac{c}{\lambda}\sinh\lambda(x-\alpha)\cosh\lambda(\beta-t) \quad \text{in } (x, \beta),$$

where $c = \text{const.}$

Let $\varphi = \cosh\lambda(t-\alpha)$ in (α, x) and $\varphi = \lambda^{-1}\sinh\lambda(\beta-t)$ in (x, β). Then we obtain the inequality

$$(21) \qquad \frac{\lambda\cosh\lambda(\beta-\alpha)}{\cosh\lambda(x-\alpha)\sinh\lambda(\beta-x)}|h(x)|^2 \leqslant \int_\alpha^\beta (\dot{h}^2+\lambda^2 h^2)\,dt$$

which holds for $\alpha < x < \beta$ and $h \in H^0$. An equality occurs in (21) only for the function

$$h = \frac{c}{\lambda}\sinh\lambda(\beta-x)\cosh\lambda(t-\alpha) \quad \text{in } (\alpha, x\rangle$$

and

$$h = \frac{c}{\lambda}\cosh\lambda(x-\alpha)\sinh\lambda(\beta-t) \quad \text{in } (x, \beta),$$

where $c = \text{const.}$

Let $\varphi = \lambda^{-1}\sinh\lambda(t-\alpha)$ in (α, x) and $\varphi = \lambda^{-1}\sinh\lambda(\beta-t)$ in (x, β). Then the inequality

$$(22) \qquad \frac{\lambda\sinh\lambda(\beta-\alpha)}{\sinh\lambda(x-\alpha)\sinh\lambda(\beta-x)}|h(x)|^2 \leqslant \int_\alpha^\beta (\dot{h}^2+\lambda^2 h^2)\,dt$$

is valid for $\alpha < x < \beta$ and $h \in H_0 \cap H^0$. An equality occurs in (22) only for the function

$$h = \frac{c}{\lambda}\sinh\lambda(\beta-x)\sinh\lambda(t-\alpha) \quad \text{in } (\alpha, x\rangle$$

and

$$h = \frac{c}{\lambda} \sinh \lambda(x-\alpha) \sinh \lambda(\beta - t) \quad \text{in } (x, \beta),$$

where $c = \text{const}$.

The inequalities (19), (20) and (22) are discussed in [1].

EXAMPLE 3. Let $I = (0, \beta)$, $0 < \beta \leqslant \infty$, $p = 2$ and $r = t^a$ in I, where a is an arbitrary constant such that $a \neq 1$. Let x be an arbitrary point of I. Further, let λ be an arbitrary fixed constant such that $\lambda > \max(0, 1-a)$.

Let $\beta = \infty$ and let us put $\varphi = t^\lambda$ in $(0, x)$ and $\varphi = t^{1-a-\lambda}$ in (x, ∞). Then we obtain the inequality

$$(23) \qquad (2\lambda + a - 1) x^{a-1} |h(x)|^2 \leqslant \int_0^\infty (t^a \dot{h}^2 + \lambda(\lambda + a - 1) t^{a-2} h^2) dt$$

for $0 < x < \infty$. If $a < 1$, then (23) is valid for $h \in H_0$; if $a > 1$, then (23) is valid for $h \in H^0$. An equality in (23) holds only for the function $h = cx^{1-a-\lambda} t^\lambda$ in $(0, x\rangle$ and $h = cx^\lambda t^{1-a-\lambda}$ in (x, ∞), where $c = \text{const}$.

Let $\beta < \infty$ and let us put

$$\varphi = t^\lambda \text{ in } (0, x) \quad \text{and} \quad \varphi = (\lambda + a - 1)(t/\beta)^\lambda + (t/\beta)^{1-a-\lambda} \text{ in } (x, \beta).$$

Then we have the inequality

$$(24) \qquad \frac{(2\lambda + a - 1) x^{a-1}}{1 + (\lambda + a - 1)(x/\beta)^{2\lambda + a - 1}} |h(x)|^2 \leqslant \int_0^\beta (t^a \dot{h}^2 + \lambda(\lambda + a - 1) t^{a-2} h^2) dt$$

which holds for $0 < x < \beta$ and $h \in \tilde{H}$; and if $a > 1$, then $\tilde{H} = H$; if $a < 1$, then $\tilde{H} = H_0$. An equality in (24) occurs only for the function

$$h = c[(\lambda + a - 1)(x/\beta)^\lambda + \lambda (x/\beta)^{1-a-\lambda}] t^\lambda \quad \text{in } (0, x\rangle$$

and

$$h = cx^\lambda [(\lambda + a - 1)(t/\beta)^\lambda + \lambda (t/\beta)^{1-a-\lambda}] \quad \text{in } (x, \beta),$$

where $c = \text{const}$.

The inequality (24) for $a > 0$, $\lambda = 1$ and $\beta = 1$ is considered in [1].

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Reçu par la Rédaction le 10.6.1983