COLLOQUIUM MATHEMATICUM

VOL. XXXIV 1976 FASC. 2

REMARKS ON STABILITY AND SATURATED MODELS

BY

J. WIERZEJEWSKI (WROCŁAW)

In this note we give a description of consequences of stability for the existence of saturated models. In [5] Shelah gave a characterization of those cardinals in which an unstable theory T has saturated models. Our theorems complete this characterization in the case where stability is assumed.

We use the standard notation as in [3]. If $\varkappa$ is an ordinal number, then

$$^{\varkappa>}A = \{f\colon f \text{ is a function, } \mathrm{dom}(f) \in \varkappa \text{ and } \mathrm{ra}(f) \subseteq A\}.$$

T always denotes a countable complete first-order theory with equality in a language L, having an infinite model. Without loss of generality we may assume that T has an elimination of quantifiers (see [1]).

Let $\mathfrak{A}$ be a model of T and $C \subseteq A$; p is a *type* over C if p is a set of formulas of the language of $\mathrm{Th}(\mathfrak{A}, c)_{c\in C}$ such that, for every formula $\varphi \in p$,

$$\mathrm{Fr}(\varphi) \subseteq \{x_0\}$$

and, for every finite $q \subseteq p$,

$$(\mathfrak{A}, c)_{c\in C} \models \exists x_0 \bigwedge_{\varphi\in q} \varphi.$$

A type p is called a *φ-type*, where φ is a formula of L, if, for every $\psi \in p$, ψ is one of the forms $\varphi(x_0, \bar{c})$ or $\neg\varphi(x_0, \bar{c})$. The maximal φ-type included in p is denoted by $p|\varphi$. A type p over C is *complete* if, for every formula ψ of the language of $\mathrm{Th}(\mathfrak{A}, c)_{c\in C}$ with at most x_0 free, either $\psi \in p$ or $\neg\psi \in p$. Similarly for φ-types. $S(C)$ $(S_\varphi(C))$ denotes the set of all complete types (φ-types) over C. A type p over C is *realized* in $\mathfrak{A}$ if there exists $a \in A$ such that, for every $\psi \in p$, we have $(\mathfrak{A}, c)_{c\in C} \models \psi[a]$. Let $\varkappa$ be an infinite cardinal number. A model $\mathfrak{A}$ of T is *$\varkappa$-saturated* whenever, for every $C \subseteq A$ and $p \in S(C)$, if $|C| < \varkappa$, then p is realized in $\mathfrak{A}$. A model $\mathfrak{A}$ is *saturated* if it is $|A|$-saturated. T is *$\varkappa$-stable* if $|S(A)| \leqslant \varkappa$ for every $\mathfrak{A} \in \mathrm{Mod}(T)$ such that $|A| \leqslant \varkappa$. T is *stable* if it is $\varkappa$-stable for some $\varkappa \geqslant \omega$.

T is *superstable* if it is $\varkappa$-stable for every $\varkappa \geqslant 2^{\omega}$. T is *unstable* if it is not stable. Let $\mathfrak{A} \in \mathrm{Mod}(T)$ and $C \subseteq A$. $I \subseteq A$ is a *set of indiscernibles* in $\mathfrak{A}$ over C if, for every formula φ of the language of $\mathrm{Th}(\mathfrak{A}, c)_{c \in C}$ and for every $i_1, \ldots, i_n, i'_1, \ldots, i'_n \in I$ such that $i_k \neq i_l$, $i'_k \neq i'_l$ for $k \neq l$ $(n \geqslant |\mathrm{Fr}(\varphi)|)$, the following holds:

$$(\mathfrak{A}, c)_{c \in C} \models \varphi[i_1, \ldots, i_n] \leftrightarrow \varphi[i', \ldots, i'_n].$$

LEMMA 1. *Let T be stable but not superstable, and $\varkappa$ an infinite cardinal number. Then there exist formulas $\{\varphi_n(x, \bar{y}_n): 0 < n < \omega\}$, a structure $\mathfrak{A} \in \mathrm{Mod}(T)$ with $|A| = \varkappa$ and sequences $\{\bar{a}_\tau: \tau \in {}^{\omega >}\varkappa\}$, $\bar{a}_\tau \in {}^{\omega >}A$, such that*

(i) *for every $\eta \in {}^{\omega}\varkappa$,*

$$p_\eta = \{\varphi_n(x, \bar{a}_{\eta \restriction n}): 0 < n < \omega\}$$

is consistent,

(ii) *for every $\tau \in {}^{\omega >}\varkappa$ and every $\xi_0 < \xi_1 < \varkappa$,*

$$\mathfrak{A} \models \neg \exists x \big(\varphi_n(x, \bar{a}_{\tau \frown \langle \xi_0 \rangle}) \wedge \varphi_n(x, \bar{a}_{\tau \frown \langle \xi_1 \rangle})\big), \quad \textit{where } n = \mathrm{lh}(\tau) + 1.$$

The proof follows by 6.10 of [6], Compactness Theorem and Downward Skolem-Löwenheim Theorem.

LEMMA 2. *Let T be stable. If T has a saturated model of power $\varkappa$ for some $\varkappa$ such that $\omega < \varkappa < \varkappa^{\omega}$, then T is superstable.*

Proof. Suppose T is not superstable. Let $\mathfrak{B}$ be a saturated model of T of power $\varkappa$ and let $\mathfrak{A}$ be as in Lemma 1. By [2] we may assume $\mathfrak{A} \prec \mathfrak{B}$. Then, for every $\eta \in {}^{\omega}\varkappa$, p_η is realized in $\mathfrak{B}$ by a $c_\eta \in B$, and if $\eta_0, \eta_1 \in {}^{\omega}\varkappa$, $\eta_0 \neq \eta_1$, then $c_{\eta_0} \neq c_{\eta_1}$. Hence

$$|B| \geqslant |\{c_\eta: \eta \in {}^{\omega}\varkappa\}| = \varkappa^{\omega} > \varkappa$$

and we get a contradiction.

The following theorem was stated in [6]. As far as we know the proof was never published.

THEOREM 1. *Let λ be an infinite cardinal number. If T is λ-stable, then it has a saturated model of power λ.*

Proof. We consider three cases.

Case 1. $\lambda = \mathrm{cf}(\lambda)$. The proof can be found in [1].

Case 2. $\omega < \mathrm{cf}(\lambda) < \lambda$.

Let $\mathfrak{B}^*$ denote an elementary extension of $\mathfrak{B}$ of minimal cardinality and such that, for every $p \in S(B)$, p is realized in $\mathfrak{B}^*$. Define, for every ordinal number α and structure $\mathfrak{B}$, a sequence of structures $\mathfrak{B}^{(\alpha)}$ in the following way:

$$\mathfrak{B}^{(0)} = \mathfrak{B}, \quad \mathfrak{B}^{(\alpha+1)} = (\mathfrak{B}^{(\alpha)})^*,$$

$$\mathfrak{B}^{(\sigma)} = \bigcup \{\mathfrak{B}^{(\xi)}: \xi < \sigma\} \quad \text{for limit } \sigma.$$

Let $\mathfrak{A}$ be a model of T of cardinality at most λ. We define a sequence of structures in the following way:

$$\mathfrak{A}_0 = \mathfrak{A}, \quad \mathfrak{A}_{\xi+1} = \mathfrak{A}^{(|\xi+\omega|^+)} \quad \text{for } \xi < \lambda,$$
$$\mathfrak{A}_\sigma = \bigcup\{\mathfrak{A}_\xi\colon \xi < \sigma\} \text{ for limit } \sigma \leqslant \lambda, \quad \mathfrak{A}_{\lambda+1} = \mathfrak{A}_\lambda^{(\lambda^+)}.$$

It is easy to check that

(a1) $\{\mathfrak{A}_\xi\colon \xi \leqslant \lambda+1\}$ forms an elementary continuous chain of models of T.

(a2) If $\xi \leqslant \lambda$, then $|A_\xi| \leqslant \lambda$ (by λ-stability of T and the cardinality assumption on $\mathfrak{A}$).

(a3) If $\xi < \lambda$, then $\mathfrak{A}_{\xi+1}$ is $|\xi+\omega|^+$-saturated.

(a4) $\mathfrak{A}_{\lambda+1}$ is λ^+-saturated.

CLAIM. *$\mathfrak{A}_\lambda$ is a saturated model of power λ.*

Of course, $|A_\lambda| = \lambda$. Suppose $B \subseteq A_\lambda$, $|B| = \varkappa < \lambda$, $p \in S(B)$, $q \in S(A_\lambda)$, and $q \supseteq p$. By 2.5, 2.9 and 2.13 of [6], for every formula $\varphi(x, \bar{y})$ of L, $\mathrm{Rank}_\varphi(q|\varphi) < \infty$ and there is a finite $q_\varphi \subseteq q|\varphi$ such that $\mathrm{Rank}_\varphi(q|\varphi) = \mathrm{Rank}_\varphi(q_\varphi)$ (for definitions see [4] and [6]). Let

$$\tilde{q} = \bigcup\{q_\varphi\colon \varphi \text{ is a formula of } L\}.$$

By the countability of T and the assumption $\mathrm{cf}(\lambda) > \omega$, we may suppose that $\tilde{q}$ is a type over some countable $C \subseteq A_0$. By 3.4 of [6], there exists $q_{2\lambda} \in S(A_{\lambda+1})$ such that $q_{2\lambda} \supseteq q$ and, for every formula φ of L,

$$\mathrm{Rank}_\varphi(q_{2\lambda}|\varphi) = \mathrm{Rank}_\varphi(q|\varphi).$$

Now we define by induction sequences $\{q_\xi\colon \xi < 2\lambda\}$ and $\{c_\xi\colon \xi < 2\lambda\}$ such that

(b1) $\xi \leqslant \eta < 2\lambda \to q_\xi \subseteq q_\eta$, $\xi \leqslant \lambda \to q_\xi \subseteq q$ and $\xi < 2\lambda \to q_\xi \subseteq q_{2\lambda}$;

(b2) $\eta < 2\lambda \to q_\eta \in S(C \cup \{c_\xi\colon \xi < \eta\})$;

(b3) $\xi < \lambda \to c_\xi \in A_{\xi+1}$ and $\lambda \leqslant \xi < 2\lambda \to c_\xi \in A_{\lambda+1}$;

(b4) $\xi < 2\lambda \to c_\xi$ realizes q_ξ, and $\lambda \leqslant \xi < 2\lambda \to c_\xi$ realizes q (hence also p).

Let $q_0 = \tilde{q}$. By the construction of $\tilde{q}$ and (a3), there exists $a \in A_1$ which realizes $\tilde{q}$. Take $c_0 = a$. Suppose that $\{q_\xi\colon \xi < \eta < \lambda\}$ and $\{c_\xi\colon \xi < \eta < \lambda\}$ have been defined. Then take

$$q_\eta = \{\varphi \in q\colon \varphi \text{ contains only constants from } C \cup \{c_\xi\colon \xi < \eta\}\}.$$

Of course, (b1) and (b2) hold. By (a3), there exists $b \in A_{\eta+1}$ which realizes q_η. Put $c_\eta = b$. Then (b3) and (b4) hold. Let $q_\lambda = \bigcup\{q_\xi\colon \xi < \lambda\}$. By (a4), there exists $c_\lambda \in A_{\lambda+1}$ which realizes q (hence also q_λ). We complete the construction of these sequences, similarly as above, taking care of (b1)-(b4) ($q_\xi \cup q \subseteq q_{2\lambda}$, so it is a type and, by (a4), can be realized in $\mathfrak{A}_{\lambda+1}$).

In view of the choice of $\tilde{q}$ and $q_{2\lambda}$, we conclude, by (b1) and 2.5 of [6], that, for every formula φ of L and every $\xi \leqslant 2\lambda$,

$$\mathrm{Rank}_{\varphi}(q_0|\varphi) = \mathrm{Rank}_{\varphi}(q_{\xi}|\varphi).$$

So, by 5.7 of [6], $I = \{c_{\xi}\colon \xi < 2\lambda\}$ is a set of indiscernibles in $\mathfrak{A}_{\lambda+1}$ over C. By 6.13 of [6], there exists $I_0 \subseteq I$ such that $|I_0| = \varkappa = |B|$ and $I \setminus I_0$ is a set of indiscernibles in $\mathfrak{A}_{\lambda+1}$ over B. Then

$$J_1 = \{c_{\xi}\colon \lambda \leqslant \xi < 2\lambda\} \cap (I \setminus I_0) \neq 0, \qquad J_2 = (I \setminus I_0) \cap A_{\lambda} \neq 0.$$

By (b4), there is $d_1 \in J_1$ which realizes q, and hence also p. Since p is a type over B, there exists $d_2 \in J_2$ which realizes p.

Case 3. $\omega = \mathrm{cf}(\lambda) < \lambda$.

In this case $\lambda^{\omega} > \lambda$, and hence, by 6.10 of [6], T is superstable. We proceed similarly as in Case 2, but using the degree of type (see Section 6 of [6]) instead of rank.

Remark. By exactly the same method we can prove the following theorem (stated without proof in [6]):

Let T be stable and let λ be an infinite cardinal number such that $\mathrm{cf}(\lambda) > \omega$. Suppose that $\{\mathfrak{A}_{\xi}\colon \xi < \lambda\}$ is an elementary increasing chain of models of T, in which every $\mathfrak{A}_{\xi}$ is $\varkappa$-saturated. Then $\mathfrak{A} = \bigcup \{\mathfrak{A}_{\xi}\colon \xi < \lambda\}$ is $\varkappa$-saturated. Moreover, if T is superstable, we can eliminate the assumption $\mathrm{cf}(\lambda) > \omega$.

We note the well-known fact: A countable Boolean algebra has ω or 2^{ω} ultrafilters.

COROLLARY 1. *If T has a saturated model of power ω_1, then either T is ω-stable or $2^{\omega} = \omega_1$.*

THEOREM 2. *Let T be stable and let $\varkappa > \omega$. Then T is $\varkappa$-stable iff T has a saturated model of power $\varkappa$.*

Proof. The necessity follows by Theorem 1.

Sufficiency. Case 1. T is ω-stable.

In this case T is $\varkappa$-stable by 2.7 of [1].

Case 2. T is superstable but not ω-stable.

If $\varkappa \geqslant 2^{\omega}$, then, of course, T is $\varkappa$-stable. Let $\omega < \varkappa < 2^{\omega}$ and let $\mathfrak{B}$ be a saturated (and hence, by [2], universal) model of T of power $\varkappa$. Let $\mathfrak{A} \prec \mathfrak{B}$ and $|S(A)| > \omega = |A|$. By the above-mentioned fact, $|S(A)| = 2^{\omega}$, and hence $\varkappa = |B| \geqslant 2^{\omega}$. We get a contradiction.

Case 3. T is stable but not superstable.

By Lemma 2, $\varkappa^{\omega} = \varkappa$ and, by 2.13 of [6], T is $\varkappa$-stable.

As a corollary we obtain a natural characterization of theories having saturated model for every uncountable power.

COROLLARY 2. (i) *T has a saturated model for every uncountable power iff T is $\varkappa$-stable for every $\varkappa > \omega$.*

(ii) *T has a saturated model for every power $\varkappa \geqslant 2^{\omega}$ iff T is superstable.*

The proof follows by Theorems 1 and 2 and by the following result of Shelah [5]: Let T be unstable and let $\varkappa > \omega$. Then T has a saturated model of power $\varkappa$ iff

$$\varkappa = \sum_{\lambda<\varkappa} \varkappa^{\lambda}.$$

The next proposition shows that we cannot generalize this characterization for $\varkappa \geqslant \omega$.

PROPOSITION. (i) *There is a superstable theory T_1 which is not ω-stable but has a countable saturated model.*

(ii) *There is a superstable theory T_2 which has not a countable saturated model.*

Proof. We shall describe the examples, but shall not prove the required properties.

(i) Let $\mathfrak{A} = \langle \omega \setminus \{0\}, R_p \rangle_{p \in P}$, where P is the set of all prime natural numbers, and R_p are binary relations defined in the following way:

$$n\, R_p\, m \text{ iff } \forall r \leqslant p \ \big(r \in P \to (p \text{ divides } n \leftrightarrow p \text{ divides } m)\big).$$

$T_1 = \mathrm{Th}(\mathfrak{A})$ satisfies (i).

(ii) Let $\mathfrak{B}_1 = \langle {}^{\omega}2, R_i \rangle_{i \in \omega}$, where R_i are unary relations defined in the following way: $R_i(f)$ iff $f(i) = 0$. Thus $T_2 = \mathrm{Th}(\mathfrak{B}_1)$ satisfies (ii).

Remark. If in the definition of $\varkappa$-stable theories we admit also finite cardinals (i.e., if we say T is *$\varkappa$-stable* whenever, for every $\mathfrak{A} \in \mathrm{Mod}(T)$ and every $C \subseteq A$ such that $|C| \leqslant \varkappa$, we have $|S(C)| \leqslant \varkappa + \omega$), we get a somewhat artificial characterization:

T has a saturated model for every infinite power iff T is $\varkappa$-stable for all $\varkappa \neq \omega$.

REFERENCES

[1] M. Morely, *Categoricity in power*, Transactions of the American Mathematical Society 114 (1965), p. 514-538.

[2] — and R. L. Vaught, *Homogeneous universal models*, Mathematica Scandinavica 11 (1962), p. 37-57.

[3] G. E. Sacks, *Saturated model theory*, 1972.

[4] S. Shelah, *Stable theories*, Israel Journal of Mathematics 7 (1969), p. 187-202.

[5] — *Finite diagram stable in power*, Annals of Mathematical Logic 2 (1970), p. 69-118.

[6] — *Stability, the f.c.p. and superstability*, ibidem 3 (1971), p. 271-362.

INSTITUTE OF MATHEMATICS
OF THE POLISH ACADEMY OF SCIENCES
WROCŁAW

Reçu par la Rédaction le 1. 3. 1974