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ON BOREL STRUCTURES

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1. Summary. The purpose of this paper is to record certain observations about Borel structures. After giving the relevant definitions in Section 2, we show in Section 3 that every countably generated Borel space has a minimal generator. In Section 4 we show that the intersection of two separable Borel structures need not be separable. In Section 5 we show that there are atomless Borel structures on any uncountable set. Finally, in Section 6 we give some applications of a theorem of Blackwell and Mackey, after a precise statement of the theorem.

2. Preliminaries. Our terminology is that of Mackey [3] (see Sections 1 and 2). Let X be any non-empty set and $\boldsymbol{B}$ a σ-algebra of subsets of X. $\boldsymbol{B}$ is called a *Borel structure* for X and $(X, \boldsymbol{B})$ — a *Borel space*. As usually, if there is no fear of confusion, we shall identify X with $(X, \boldsymbol{B})$. A family $\boldsymbol{G} \subset \boldsymbol{B}$ is called a *generator* for $\boldsymbol{B}$ if the smallest Borel structure on X containing $\boldsymbol{G}$ coincides with $\boldsymbol{B}$. A generator $\boldsymbol{G}$ is called *minimal* if no proper subfamily of it is a generator for $\boldsymbol{B}$. $\boldsymbol{B}$ is called *countably generated* if there is a countable generator. An *atom* of $\boldsymbol{B}$ is a set A in $\boldsymbol{B}$ which is not empty and such that no non-empty proper subset of A is in $\boldsymbol{B}$. $\boldsymbol{B}$ is called *separable* if it is countably generated and contains singletons. An *isomorphism* between two Borel spaces is a one-to-one bimeasurable map of one onto the other. As in Mackey, the relativized concepts can also be defined. If $(X, \boldsymbol{B})$ is a Borel space and Y is a subset of X, then the relativized Borel structure on Y is denoted by $\boldsymbol{B}_Y$.

2^ω denotes the unilateral countable product of the two point space $\{0, 1\}$. The *Borel structure* $\boldsymbol{A}$ on 2^ω is the product of discrete Borel structures on component spaces. If $\{G_n, n \geqslant 1\}$ is a generator for a separable Borel space $(X, \boldsymbol{B})$, then the *Marczewski function* [4] defined as

$$f(x) = \{\chi_{G_n}(x);\ n \geqslant 1\}$$

is a Borel isomorphism between X and the range of f in 2^ω. $(I, \boldsymbol{B})$ always denotes the closed unit interval with its usual Borel structure.

3. Minimal generators. In an oral communication Dr. D. Basu has raised the question whether the usual Borel structure $\boldsymbol{B}$ on I has a minimal generator. This question is answered in the affirmative by the following theorem:

THEOREM 1. *Any countably generated Borel space has a minimal countable generator.*

Proof. Let $(X, \boldsymbol{B})$ be any countably generated Borel space. Clearly, it suffices to consider the case where X is infinite.

We start with observing that the Borel structure $\boldsymbol{A}$ on 2^ω has a minimal generator. Take A_n to be all those points of 2^ω whose n-th coordinate is 1. Clearly, $\{A_n, n \geqslant 1\}$ is a generator for $\boldsymbol{A}$ and the removal of A_k results in a Borel structure which cannot distinguish points differing only in the k-th coordinate.

Define a_n for $n \geqslant 1$ to be that point of 2^ω which has only the n-th coordinate zero and a_0 to be the point which has zero in no coordinate. Observe that if Z is any subset of 2^ω containing $\{a_n, n \geqslant 0\}$, then $\boldsymbol{A}_Z$ has a minimal generator. Enough to take $B_n = A_n \cap Z$; where $n \geqslant 1$ and A_n are as described above.

Now take any separable space $(X, \boldsymbol{B})$. Then the Marczewski function brings an isomorphism between $(X, \boldsymbol{B})$ and a subset Z of 2^ω. By suitable altering the map on a countable subset of X we can assume that Z contains the points $\{a_n, n \geqslant 0\}$ described above. Since the property of possessing a minimal generator is an isomorphic invariant, $(X, \boldsymbol{B})$ has a minimal generator.

Finally, if $(X, \boldsymbol{B})$ is any countably generated Borel space, then it is in an obvious way structure isomorphic to a separable space. (Look at the space $\tilde{X}$ of atoms of $\boldsymbol{B}$ with the natural Borel structure $\tilde{\boldsymbol{B}}$.) Hence it has a minimal generator. This completes the proof of the theorem.

Remark 1. If $(X, \boldsymbol{B})$ is any separable Borel space and G any generator for B, then G contains a countable subfamily which is also a generator for $\boldsymbol{B}$. However, G need not contain a subfamily which is a minimal generator. For instance, take $(I, \boldsymbol{B})$ with $G = \{[0, a); 0 < a \leqslant 1\}$.

Remark 2. If $(X, \boldsymbol{B})$ is any Borel space and G any generator for $\boldsymbol{B}$ and $Y \subset X$, then G_Y is a generator for $\boldsymbol{B}_Y$. However, if G is a minimal generator for $\boldsymbol{B}$, then G_Y need not be minimal generator for $\boldsymbol{B}_Y$.

Remark 3. If $\{(X_a, \boldsymbol{B}_a), a \in \Delta\}$ is a collection of separable Borel spaces, then their product $(X, \boldsymbol{B})$ is separable iff all but countable number of X_a consist of a single point. However, if each $\boldsymbol{B}_a$ has a minimal generator (though not separable), then $\boldsymbol{B}$ also has a minimal generator. To see this fix any minimal generator in the coordinate spaces and look at the one-dimensional cylinder subsets of X whose base lies in the fixed minimal generator.

Remark 4. We do not know of any Borel structure without a minimal generator (**P685**). We have two possible candidates for this purpose. The first is the Borel structure on I generated by its analytic sets. It is known [5] that it is not countably generated. The second is also a Borel structure for I obtained as follows: Fix a non-Borel set $M \subset I$ and look at the σ-algebra $\boldsymbol{C}$ consisting of all those usual Borel subsets of I which are either disjoint with M or containing M. Observe that if M^c does not contain a perfect set, then $\boldsymbol{C}$ can be very simply characterized as all countable subsets of M^c and cocountable subsets of I containing M and has got a minimal generator. For general M we do not know the answer.

4. Separable Borel structures.

THEOREM 2. *There are two countably generated Borel structures on I whose intersection is not countably generated.*

Proof. Let $\boldsymbol{B}$ be the usual Borel structure for I. Fix a non-Borel set M in I. Let $\boldsymbol{B}_0$ be the structure on I generated by M and $\boldsymbol{B}_{M^c}$. The intersection of these two is the $\boldsymbol{C}$ of Remark 4, which is clearly not countably generated. However, these two structures $\boldsymbol{B}$ and $\boldsymbol{B}_0$ are countably generated.

Remark 5. In fact, the above proof shows something more. Given any separable Borel structure to I we can find a countably generated Borel structure whose intersection with the given one is not countably generated.

Remark 6. It has been remarked by Dr. J. K. Ghosh that we can, in fact, get two substructures of the usual Borel structure of the real line, both of which are countably generated while their intersection is not. For instance, consider the structure $\boldsymbol{L}$ consisting of all usual Borel sets invariant under translation by unit, and the structure $\boldsymbol{M}$ consisting of all usual Borel sets invariant under translation by i, where i is a fixed irrational number.

Observe that in $\boldsymbol{B}_0$ of the above proof no singleton subset of M is available and hence $\boldsymbol{B}_0$ is not separable. In fact, we can give two separable structures to I with the above property. However, to do this we need a lemma due to Halmos [1], section 7. The idea of the proof is essentially the same as that of [1]. Since the proof is simple, we shall give a complete proof here. Without explicit mention, the axiom of choice has been assumed below.

LEMMA 1. *There is a one-to-one map f of I onto I such that*

(i) $f = f^{-1}$;

(ii) *if A and A^c are uncountable Borel subsets of I, then $f^{-1}(A)$ is not Borel.*

Proof. Let $\Omega_{\mathfrak{c}}$ be the first ordinal corresponding to the cardinal $\mathfrak{c}$. Let $\{A_a; 1 \leqslant a < \Omega_{\mathfrak{c}}\}$ be an enumeration of the uncountable Borel subsets of I whose complements are also uncountable. Since every uncountable Borel set has cardinality $\mathfrak{c}$, we can associate with each $a < \Omega_{\mathfrak{c}}$ three distinct points x_a, y_a, z_a of I such that

(i) $x_a, y_a \in A_a, z_a \in A_a^c$;

(ii) $x_a, y_a, z_a \notin \bigcup_{\beta<a} \{x_\beta, y_\beta; z_\beta\}$.

Let f be the map which interchanges x_a with z_a and keeps every other point fixed. Since any A_a contains uncountable number of A_β's and since y_a's are kept fixed by f, we conclude that uncountable number of points of A_a are kept fixed by f. Clearly, no A_a is kept invariant by f. These two facts imply that if $f^{-1}(A_a)$ is Borel, then $f^{-1}(A_a) \cap A_a$ can neither be countable nor uncountable. Hence $f^{-1}(A_a)$ is not Borel as desired.

THEOREM 3. *There are two separable Borel structures to I whose intersection is not separable.*

Proof. Let $\boldsymbol{B}$ be the usual structure and $\boldsymbol{B}_0 = f^{-1}(\boldsymbol{B})$, where f is any map satisfying the conditions of Lemma 1. Clearly, $\boldsymbol{B}$ and $\boldsymbol{B}_0$ are separable, but their intersection is countable-cocountable structure.

Remark 7. The above proof shows the following stronger fact: Given any separable Borel structure to I such that every set in this structure is either countable or has cardinality $\mathfrak{c}$, there is another such a structure whose intersection with the given one is the countable-cocountable structure.

Remark 8. We do not know whether the function f of Lemma 1 can be chosen so as to satisfy the further condition:

(iii) for every Borel set $A, f^{-1}(A)$ is in the Borel structure on I generated by analytic sets (**P686**).

Remark 9. We have given above two separable Borel structures for I whose intersection does not contain any separable structure. We do not know whether it is possible to give two separable structures whose intersection is not separable but contains a separable structure (**P687**).

Remark 10. There is an alternative way, which seems to be elegant, of proving Theorem 3. See Remark 14 of section 6.

5. Atomless structures. The usual example of a Borel space which is atomless is I^I with the product of the usual Borel structures. One can ask whether there is such a structure on I itself. We shall answer this in the following theorem:

THEOREM 4. *For any uncountable set X, there is an atomless Borel structure.*

Proof. Let $\aleph_1$ be the first uncountable cardinal number. Equip $X^{\aleph_1}$ with the product of discrete (in fact, any nice) Borel structures on X. Fix any $p \in X$. Let $X_0 \subset X$ be the set consisting of all those points which have p in all but finite number of coordinate places. Then observe that X_0 and X have the same cardinality and that the relativized structure on X_0 is atomless and can be carried over to X. This proves the theorem.

Remark 11. Clearly, a Borel structure is atomless iff every non-empty Borel set contains two disjoint non-empty Borel sets. It is interesting to observe that this is also equivalent to saying that every non-empty Borel set contains $\aleph_1$ disjoint non-empty Borel sets.

Remark 12. The above theorem says, for instance, that the real line has an atomless Borel structure. In fact, one can find such a structure which is translation invariant. If X denotes the real line in the proof of Theorem 4, observe that X and X_0 are vector spaces over rationals and have the same dimension $\mathfrak{c}$, and hence there is a one-to-one additive map on X onto X_0. Since the structure on X_0 is translation invariant, the structure on X brought by any 1-1 additive function will also be translation invariant.

6. A theorem of Blackwell-Mackey. For any Borel space $(X, \boldsymbol{B})$ let $(X^2, \boldsymbol{B}^2)$ denote the product of it with itself. Call a set $A \in \boldsymbol{B}^2$ *symmetric* iff $(x, y) \in A$ implies $(y, x) \in A$. Clearly, a rectangle is symmetric iff both sides are the same. The symmetric sets form a Borel structure on X^2. Mr. K. Viswanath has raised the question whether the symmetric structure for I^2 is generated by the symmetric rectangles. In this section we see how Blackwell-Mackey theorem will be instructive in answering this question in the affirmative.

Call a separable space $(X, \boldsymbol{B})$ *analytic* if it is isomorphic to an analytic subset of I. By using the first principle of separation for analytic sets, Blackwell [2], section 4, and Mackey [3], section 4, have explicitly noticed the following theorem:

If $(X, \boldsymbol{B})$ is an analytic space and $\boldsymbol{C}, \boldsymbol{D} \subset \boldsymbol{B}$ are countably generated sub-σ-algebras with the same atoms, then $\boldsymbol{C} = \boldsymbol{D}$.

Remark 13. Since $(I, \boldsymbol{B})$ is an analytic space, it follows, in particular, that any enlargement of $\boldsymbol{B}$ cannot be Borel isomorphic to $\boldsymbol{B}$ (compare with [1], p. 628, lines 3, 4).

Remark 14. Take an uncountable Borel subset B and an analytic non-Borel subset A of I with relative structures $\boldsymbol{B}_B$ and $\boldsymbol{B}_A$, respectively. Let f and g be any one to one functions on I onto B and A, respectively. Let $\boldsymbol{B}_f = f^{-1}(\boldsymbol{B}_B)$ and $\boldsymbol{B}_g = g^{-1}(\boldsymbol{B}_A)$. Then, clearly, $\boldsymbol{B}_f$ and $\boldsymbol{B}_g$ are separable Borel structures for I whose intersection, by the theorem quoted above, is not separable.

THEOREM 5. *For any Borel space $(X, \boldsymbol{B})$ the symmetric structure on X^2 is generated by symmetric rectangles.*

Proof. If $X = I$ and $\boldsymbol{B}$ is the usual structure, then the theorem is a consequence of the Blackwell-Mackey theorem. If $X \subset I$ and $\boldsymbol{B}$ is the relativized Borel σ-algebra of I, then the result is still clear, because the symmetric structure on X^2 is that restricted from I^2 and the structure on X^2 generated by symmetric rectangles is the restriction to X^2 of the corresponding structure for I^2. Consequently, the result is true for any separable space $(X, \boldsymbol{B})$ in view of the Marczewski function. Because of the structure isomorphism the result is true for any countably generated Borel space $(X, \boldsymbol{B})$.

If $(X, \boldsymbol{B})$ is any Borel space, observe that the symmetric structure on X^2 always contains the structure generated by symmetric rectangles. To the converse, take any symmetric set $A \in \boldsymbol{B}^2$; we show that it is available in the structure generated by symmetric rectangles. Clearly, there is a countably generated $\boldsymbol{B}_0$ for X contained in $\boldsymbol{B}$ such that $A \in \boldsymbol{B}_0^2$. Now applying the conclusion of section 5 we get the result.

Remark 15. The above theorem can also be proved by transfinite induction.

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Added in proof. Following the argument of [5] it can be shown that P686 has a negative solution.

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