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A CANONICAL FORM FOR A SYSTEM OF QUADRATIC FUNCTIONAL EQUATIONS

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Throughout this note we use the Einstein summation convention, i.e. summation is implied on repeated indices $i, j, k, \ldots$ from 1 to N, and on the repeated index a from 0 to N.

We consider the system of functional equations given by

$$g_k(x+y) = \Gamma_k^{ij} g_i(x) g_j(y), \tag{1}$$

where g_i: $F \to C$ ($F = R$ or $F = C$) are linearly independent functions and $\Gamma_k = (\Gamma_k^{ij})$ are $(N \times N)$-matrices over C.

System (1) is equivalent to (see [3]) the system

$$h(x+y) = A^{ij} f_i(x) g_j(y), \tag{2}$$

where h, f_i, g_i: $F \to C$, and (A^{ij}) is a matrix over C, if we assume that $\det(A^{ij}) \neq 0$.

Both systems (1) and (2) have been studied quite extensively by various authors: see [2] and [4] for examples of (1), and [1] for examples of (2); see also [3] for further references to both.

In the following we show that the existence of a C^2-solution of (1) implies a canonical form for this system. This canonical form reduces (1) to the system of equations

$$g_k(x+y) = \sum_{r=0}^{k} g_r(x) g_{k-r}(y), \tag{i}$$

whose general solution was found in [1]. In effect, this shows that *if a C^2-solution of* (1) *is known, then the general solution is also known, formed by linear combinations of solutions to* (i).

Since various weak regularity conditions imply C^2-differentiability ([1] and [3]), our hypotheses may be easily weakened.

Accordingly, we assume that $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_N$ is a C^2-solution of (1), so we have

$$(3) \qquad \bar{g}_k(x+y) = \Gamma_k^{ij}\bar{g}_i(x)\bar{g}_j(y).$$

If $F = C$, then the $\bar{g}_i$'s have all derivatives, since they are analytic. If $F = R$ and the a-th derivatives of the $\bar{g}_i$'s exist, then by differentiating (3) once with respect to x and $a-1$ times with respect to y, we obtain, with $x = 0$,

$$\bar{g}_k^{(a)}(y) = \Gamma_k^{ij}\bar{g}_i^{(1)}(0)\bar{g}_j^{(a-1)}(y), \qquad \text{where } \bar{g}_k^{(\beta)}(y) = \frac{d^\beta \bar{g}}{dy^\beta}(y).$$

Hence the $(a+1)$-st derivatives of the $\bar{g}_i$'s exist, and so, by induction, all derivatives of the $\bar{g}_i$'s exist. Thus a C^2-solution of (1) is automatically a C^∞-solution of (1).

LEMMA. *If $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_N$ is a C^2-solution of* (1), *then there exists a unique linear N-th order differential equation with constant coefficients simultaneously satisfied by all $\bar{g}_i$, assumed linearly independent.*

Proof. Differentiating (3) a times with respect to x yields, with $x = 0$,

$$(4) \qquad \bar{g}_k^{(a)}(y) = \Gamma_k^{ij}a_i^a\bar{g}_j(y), \qquad \text{where } a_i^a \stackrel{\text{df}}{=} \bar{g}_i^{(a)}(0).$$

The $(N\times(N+1))$-matrix (a_i^a) takes some non-zero $(N+1)$-tuple $(\nu_0, \nu_1, \ldots, \nu_N)$ into $(0, 0, \ldots, 0)$, i.e. $a_i^a\nu_a = 0$ for all $i = 1, 2, \ldots, N$. Thus, from (4)

$$(5) \qquad \nu_a\bar{g}_k^{(a)}(y) = 0.$$

Any two such $(N+1)$-tuples are proportional, and $\nu_N \neq 0$, since otherwise the N linearly independent functions $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_N$ would satisfy a linear ordinary differential equation of order less than N, with constant coefficients, which is impossible.

Using this lemma, we now derive a canonical form for system (1) which depends only on multiplicities of distinct roots of the complementary polynomial of the differential equation (5).

THEOREM. *Let $\bar{g}_1, \ldots, \bar{g}_N$ be a C^2-solution of* (1) *with each $\bar{g}_i$ satisfying equation* (5). *If $\bar{g}_i$'s are linearly independent and the polynomial equation $\nu_aZ^a = 0$ has r distinct (complex) roots $a_1, a_2, \ldots, a_r$ with multiplicities $s_1, s_2, \ldots, s_r$, respectively, then there exists a non-singular matrix $q = (q_j^i)$ such that the matrices $\Lambda_k = (\Lambda_k^{ij})$, defined by*

$$(6) \qquad \Lambda_k^{ij} = (q^{-1})_k^n q_l^i q_m^j \Gamma_n^{lm},$$

have the canonical form

$$\Lambda_k = \operatorname{diag}(\Lambda_{k1}, \Lambda_{k2}, \ldots, \Lambda_{kr}),$$

where

(i) Λ_{ki} *is an* $(s_i \times s_i)$*-matrix,*

(ii) $\Lambda_{ki} = 0$ *unless* $p_i+1 \leqslant k \leqslant p_{i+1}$,

(iii) *if* $p_i+1 \leqslant k \leqslant p_{i+1}$, *then*

$$
(7) \qquad \Lambda_{ki} = \begin{pmatrix}
0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0\\
\vdots & & & \binom{j-1}{j-2} & 0 & & & \vdots\\
\vdots & 0 & \iddots & 0 & & & & \vdots\\
0 & \binom{j-1}{1} & \iddots & & & & & \vdots\\
1 & 0 & & & & & & \vdots\\
0 & & & & & & & \vdots\\
\vdots & & & & & & & \vdots\\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix},
$$

$p_0, \ldots, p_r$ *being integers defined by*

$$p_0 = 0, \quad p_j = \sum_{i=1}^{j} s_i \quad \textit{for } j = 1, \ldots, r, \textit{ and } j = k-p_i.$$

Thus, the transformed functions $\tilde{g}_k(x) \overset{\mathrm{df}}{=} (q^{-1})^i_k g_i(x)$, *where* $g_1, g_2, \ldots, g_N$ *is any solution of* (1), *satisfy the canonical system*

$$(8) \qquad \tilde{g}_k(x+y) = \Lambda^{ij}_k \tilde{g}_i(x)\tilde{g}_j(y).$$

Proof. Using the binomial theorem, it is easily verified that the s functions given by $f_i(x) = x^{i-1}e^{ax}$, $i = 1, 2, \ldots, s$, $a \in C$, satisfy

$$f_k(x+y) = \sum_{i,j=1}^{s} \Omega^{ij}_k f_i(x) f_j(y),$$

where $\Omega_k = (\Omega^{ij}_k)$ is an $(s \times s)$-matrix of form (7) with j replaced by k.

Applying this result to the r blocks of functions given by

$$h_1(x) = e^{a_1 x},\ h_2(x) = xe^{a_1 x}, \ldots, h_{p_1}(x) = x^{s_1-1}e^{a_1 x},$$
$$h_{p_1+1}(x) = e^{a_2 x},\ h_{p_1+2}(x) = xe^{a_2 x}, \ldots, h_{p_2}(x) = x^{s_2-1}e^{a_2 x}, \ldots,$$
$$h_{p_{r-1}+1}(x) = e^{a_r x},\ h_{p_{r-1}+2}(x) = xe^{a_r x}, \ldots, h_N(x) = x^{s_r-1}e^{a_r x},$$

we infer that the functions $h_1, h_2, \ldots, h_N$ satisfy the equations

$$(9) \qquad h_k(x+y) = \Lambda^{ij}_k h_i(x) h_j(y),$$

where $\Lambda_k = (\Lambda^{ij}_k)$ are the canonical matrices described above.

But the h_i's are N linearly independent solutions of (5) as the $\bar{g}_i$'s are. Therefore, there exists a non-singular matrix $q = (q^i_j)$ such that $\bar{g}_k(x) = q^i_k h_i(x)$. Using this relation and equations (3) and (9) we obtain

$$[A^{ij}_k - (q^{-1})^n_k q^i_1 q^j_m \Gamma^{1m}_n] h_i(x) h_j(y) = 0$$

which, since the h_i's are linearly independent, yields equation (6).

Equation (8) says that, for $p_i+1 \leqslant k \leqslant p_{i+1}$, we have

$$\tilde{g}_k(x+y) = \sum_{s=0}^{k-p_i-1} \binom{k-p_i-1}{s} \tilde{g}_{k-s}(x)\tilde{g}_{p_i+s+1}(y). \tag{10}$$

By making the transformation

$$\hat{g}_s(x) = \frac{\tilde{g}_s(x)}{(s-p_i-1)!} \quad \text{for } p_i+1 \leqslant s \leqslant p_{i+1},$$

(10) becomes

$$\hat{g}_s(x+y) = \sum_{s=0}^{k-p_i-1} \hat{g}_{k-s}(x)\hat{g}_{p_i+1+s}(y) \tag{11}$$

which is the system of equations for the composition of r blocks of Poisson distributions. Equations (11) have been thoroughly treated elsewhere (see [1] and [2]).

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