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SOME RESULTS ON FIXED POINTS AND THEIR CONTINUITY

BY

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1. Introduction. Luxemburg [6] has proved contraction mapping theorem in a generalized metric space. Diaz and Margolis [2] have given fixed-point theorem in this setting. Fraser and Nadler [3] discussed the continuity of fixed points for contraction maps in a metric space having different metrics. The type of maps considered in Theorem 2.1 of this paper was first considered by Kannan [4] who had proved fixed-point theorems for a pair of such maps in a complete metric space, and he [5] had also shown that contraction maps and those of type considered by him are independent in the sense that neither implies the other. The purpose of the paper is to prove theorems on simultaneous fixed points in a generalized metric space, and to discuss their continuity in a metric space having different metrics.

2. Fixed-point theorems.

THEOREM 2.1. *Let $\{T_n\}$ be a sequence of maps, each mapping a generalized complete metric space (X, d) into itself, such that*

$$d\big(T_i(x), T_j(y)\big) \leqslant \beta\big[d\big(x, T_i(x)\big) + d\big(y, T_j(y)\big)\big],$$

where $0 < \beta < 1/2$, and $d(x, y) < \infty$. If $x_0 \in X$ and $x_n = T_n(x_{n-1})$, $n = 1, 2, \ldots$, then either (1) *$d(x_i, x_{i+1}) = \infty$ for every integer i or* (2) *$\{x_n\}$ d-converges to a common fixed point of $\{T_n\}$.*

Proof. Suppose (1) does not hold. Then there is a positive integer N such that $d(x_N, x_{N+1}) < \infty$. Now,

$$\begin{aligned} d(x_{N+1}, x_{N+2}) &= d\big(T_{N+1}(x_N), T_{N+2}(x_{N+1})\big) \\ &\leqslant \beta[d(x_N, x_{N+1}) + d(x_{N+1}, x_{N+2})], \end{aligned}$$

whence

$$d(x_{N+1}, x_{N+2}) \leqslant \frac{\beta}{1-\beta} d(x_N, x_{N+1}) < \infty.$$

Similarly,

$$d(x_{N+2}, x_{N+3}) \leqslant \frac{\beta}{1-\beta} d(x_{N+1}, x_{N+2}) \leqslant \left(\frac{\beta}{1-\beta}\right)^2 d(x_N, x_{N+1}) < \infty.$$

and, in general,

$$d(x_{N+i}, x_{N+i+1}) \leqslant \left(\frac{\beta}{1-\beta}\right)^i d(x_N, x_{N+1}) < \infty$$

for every integer i.

Hence, for $n > N$, we have

$$d(x_n, x_{n+1}) \leqslant \left(\frac{\beta}{1-\beta}\right)^{n-N} d(x_N, x_{N+1}) < \infty,$$

and

$$\begin{aligned} d(x_n, x_{n+l}) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+l-1}, x_{n+l}) \\ &\leqslant (r^{n-N} + r^{n+1-N} + \ldots + r^{n+l-1-N}) d(x_N, x_{N+1}) \\ &= r^{n-N} \frac{1-r^l}{1-r} d(x_N, x_{N+1}), \end{aligned}$$

where $r = \beta/(1-\beta) < 1$.

Since $0 < r < 1$, $\{x_n\}$ is d-Cauchy in X. Since (X, d) is complete, $\{x_n\}$ d-converges to some point x in X. Let T_{n_0} be a member of $\{T_n\}$. Then, taking $n > N$, we have

$$\begin{aligned} d(x, T_{n_0}(x)) &\leqslant d(x, x_n) + d(x_n, T_{n_0}(x)) \\ &= d(x, x_n) + d(T_n(x_{n-1}), T_{n_0}(x)) \\ &\leqslant d(x, x_n) + \beta[d(x_{n-1}, x_n) + d(x, T_{n_0}(x))], \end{aligned}$$

whence

$$(1-\beta) d(x, T_{n_0}(x)) \leqslant d(x, x_n) + \beta d(x_{n-1}, x_n).$$

Letting $n \to \infty$, we have $d(x, T_{n_0}(x)) = 0$, i. e., $x = T_{n_0}(x)$. Thus x is a common fixed point of $\{T_n\}$.

We remark that Theorem 2.1 remains true in a complete metric space, where alternative (1) does not arise, and common fixed point x so derived is unique, because if $y \neq x$ is another such a point, then

$$0 < d(y, x) = d(T_m(y), T_n(x)) \leqslant \beta[d(y, T_m(y)) + d(x, T_n(x))] = 0,$$

a contradiction.

THEOREM 2.2. *Let $\{T_n\}$ be a sequence of maps, each mapping a generalized complete metric space (X, d) into itself, such that*

(i) *for any two maps T_i, T_j and for all x, y $(y \neq x)$ in X with $d(x, y) < \infty$, we have $d(T_i(x), T_j(y)) \leqslant \lambda d(x, y)$, where $0 < \lambda < 1$,*

(ii) *there is a point* x_0 *in* X *such that any two consecutive members of* $\{x_n\}$, *where* $x_n = T_n(x_{n-1})$, $n = 1, 2, \ldots$, *are distinct.*

Then either (1) $d(x_i, x_{i+1}) = \infty$ *for every integer* i *or* (2) $\{x_n\}$ *d-converges to a common fixed point of* $\{T_n\}$.

Proof. Suppose (1) does not hold. Then there is a positive integer N such that $d(x_N, x_{N+1}) < \infty$, and we proceed as in the proof of Theorem 2.1 to show that $\{x_n\}$ is d-Cauchy in X. Since (X, d) is complete, $\{x_n\}$ d-converges to a point x in X. We now show that x is a common fixed point of $\{T_n\}$. Let T_m be a member of $\{T_n\}$. Then taking $n > N$, we have

$$\begin{aligned} d\big(x, T_m(x)\big) \leqslant d(x, x_n) + d\big(x_n, T_m(x)\big) &= d(x, x_n) + d\big(T_n(x_{n-1}), T_m(x)\big) \\ &\leqslant d(x, x_n) + \lambda d(x_{n-1}, x). \end{aligned}$$

Letting $n \to \infty$, we have $d\big(x, T_m(x)\big) = 0$, i. e., $x = T_m(x)$. Hence x is a common fixed point of $\{T_n\}$.

As before, we observe that Theorem 2.2 remains valid in a complete metric space, where alternative (1) does not arise, and the common fixed point is unique.

The necessity of condition (ii) of Theorem 2.2 is shown in the following example:

Example 2.1. Let X consist of two distinct elements x_1 and x_2, and let (X, d) be a metric space with $d(x_1, x_2) < \infty$. Define T_1 as $T_1(x_1) = x_2$ and $T_1(x_2) = x_1$. Take T_2 to be the identity map and $T_i = T_{i-2}$ for $i = 3, 4, 5, \ldots$ All conditions, except (ii), of Theorem 2.2 are satisfied and $\{T_i\}$ have no common fixed point.

THEOREM 2.3. *Suppose* T_i $(i = 1, 2, \ldots, N)$ *are finite number of maps, each mapping a metric space* (X, d) *into itself, such that each* T_i *is continuous at a point* u, *and there is a point* x_0 *in* X *such that the sequence* $\{x_n\colon x_n = T_n(x_{n-1})\}$, *where* $T_{mN+r} = T_r$ *for* $r = 1, 2, \ldots, N$ *and* $m = 0, 1, 2, \ldots$, *converges to* u; *then* u *is a common fixed point of* T_i, $i = 1, 2, \ldots, N$.

Proof. Let T_r be any member. Now

$$\begin{aligned} (1) \qquad d\big(T_r(u), u\big) &\leqslant d\big(T_r(u), x_{mN+r}\big) + d(x_{mN+r}, u) \\ &= d\big(T_r(u), T_r(x_{mN+r-1})\big) + d(x_{mN+r}, u). \end{aligned}$$

Since $\{x_n\}$ converges to u, $\{x_{mN+r}\}_{m=1}^{\infty}$ and $\{x_{mN+r-1}\}_{m=1}^{\infty}$ also converges to u. By continuity of T_r at u, we have

$$\lim_{m\to\infty} T_r(x_{mN+r-1}) = T_r(u).$$

As $m \to \infty$, (1) gives $T_r(u) = u$. So u is a common fixed point of T_i $(i = 1, 2, \ldots, N)$.

The extension of Theorem 2.3 over a sequence of distinct maps is not possible. This is evident from the following example:

Example 2.2. Let $X = [0, 1]$ with the usual metric, and $T_i(x) = 1/i$ for all $x \in [0, 1]$ and $i = 1, 2, \ldots$ Then, for every point x_0 in $[0, 1]$, the sequence $\{x_n: x_n = T_n(x_{n-1})\}$ converges to zero which is a point of continuity of each T_i. But T_i $(i = 1, 2, \ldots)$ have no common fixed point.

3. Continuity of fixed points.

THEOREM 3.1. *Let (X, d) be a metric space, let $T_i: X \to X$ be a map with a fixed point u_i for $i = 0, 1, 2, \ldots$, and let $\{T_i\}_{i=1}^{\infty}$ converge uniformly to T_0. If*

$$d\big(T_0(x), T_0(y)\big) \leqslant \beta\big[d\big(x, T_0(x)\big) + d\big(y, T_0(y)\big)\big],$$

where β is any positive number and $x, y \in X$, then $\{u_i\}$ converges to u_0.

Proof. Let be given $\varepsilon > 0$. By the uniform convergence of $\{T_i\}$ to T_0, there is an index N such that $d\big(T_i(x), T_0(x)\big) < \varepsilon/(1+\beta)$ for all x in X and $i \geqslant N$. Then

$$\begin{aligned} d(u_i, u_0) &= d\big(T_i(u_i), T_0(u_0)\big) \leqslant d\big(T_i(u_i), T_0(u_i)\big) + d\big(T_0(u_i), T_0(u_0)\big) \\ &\leqslant d\big(T_i(u_i), T_0(u_i)\big) + \beta\big[d\big(u_i, T_0(u_i)\big) + d\big(u_0, T_0(u_0)\big)\big] \\ &= d\big(T_i(u_i), T_0(u_i)\big) + \beta d\big(T_i(u_i), T_0(u_i)\big) \\ &= (1+\beta) d\big(T_i(u_i), T_0(u_i)\big) < (1+\beta)\frac{\varepsilon}{1+\beta} = \varepsilon, \end{aligned}$$

whenever $i \geqslant N$. Thus $\{u_i\}$ converges to u_0.

THEOREM 3.2. *Suppose (X, d_0) is a metric space and $\{d_n\}$ is a sequence of metrics converging uniformly to d_0. Let $\{T_n\}$ be a sequence of maps converging d_0-pointwise to a map T_0 with fixed point u_0 and let each T_n having fixed point u_n satisfy*

$$d_n\big(T_n(x), T_n(y)\big) \leqslant \beta\big[d_n\big(x, T_n(x)\big) + d_n\big(y, T_n(y)\big)\big],$$

where β is any positive number and $x, y \in X$. Then $\{u_n\}$ converges to u_0.

Proof. Let be given $\varepsilon > 0$. Since $\{d_n\}$ converges uniformly to d_0, and $\{T_n\}$ converges d_0-pointwise to T_0, there is an index N such that

$$|d_n(x, y) - d_0(x, y)| < \frac{\varepsilon}{2(1+\beta)}$$

and

$$d_0\big(T_n(u_0), T_0(u_0)\big) < \frac{\varepsilon}{2(1+\beta)},$$

whenever $n \geqslant N$. Thus, for $n \geqslant N$, we have

$$\begin{aligned} d_0(u_n, u_0) &= d_0\big(T_n(u_n), T_0(u_0)\big) \\ &\leqslant d_0\big(T_n(u_n), T_n(u_0)\big) + d_0\big(T_n(u_0), T_0(u_0)\big) \\ &< d_n\big(T_n(u_n), T_n(u_0)\big) + \frac{\varepsilon}{2(1+\beta)} + \frac{\varepsilon}{2(1+\beta)} \\ &\leqslant \beta\big[d_n\big(u_n, T_n(u_n)\big) + d_n\big(u_0, T_n(u_0)\big)\big] + \frac{\varepsilon}{1+\beta} \\ &< \beta d_0\big(u_0, T_n(u_0)\big) + \frac{\beta\varepsilon}{2(1+\beta)} + \frac{\varepsilon}{1+\beta} \\ &< \frac{\beta\varepsilon}{2(1+\beta)} + \frac{\beta\varepsilon}{2(1+\beta)} + \frac{\varepsilon}{1+\beta} = \varepsilon. \end{aligned}$$

This shows that $\{u_n\}$ converges to u_0.

THEOREM 3.3. *Let (X, d) be a metric space, and $T_i\colon X \to X$ be a map with a fixed point u_i such that*

$$d\big(T_i(x), T_i(y)\big) \leqslant \beta\big[d\big(x, T_i(x)\big) + d\big(y, T_i(y)\big)\big],$$

where β is any positive number and $i = 1, 2, 3, \ldots$ If $\{T_i\}$ converges pointwise to a map T_0 that maps X into itself with $T_0(u_0) = u_0$, then $\{u_i\}$ converges to u_0.

This is a corollary of Theorem 3.2 if we take $d_n = d$ for all n.

THEOREM 3.4. *Suppose (X, d_0) is a metric space, and $\{d_n\}$ is a sequence of metrics converging uniformly to d_0. Let $T_n\colon X \to X$ be a map with a fixed point u_n such that*

$$d_n\big(T_n(x), T_n(y)\big) \leqslant \beta\big[d_n\big(x, T_n(x)\big) + d_n\big(y, T_n(y)\big)\big],$$

where $0 < \beta < 1$ and $n = 1, 2, 3, \ldots$ If $\{T_n\}$ converges d_0-pointwise to a map T_0 and if u_0 is a d_0-limit point of $\{u_n\}$, then u_0 is a fixed point of T_0.

Proof. Since u_0 is a d_0-limit point of $\{u_n\}$, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ that d_0-converges to u_0. Let be given $\varepsilon > 0$. Since $\{d_n\}$ converges uniformly to d_0 and $\{T_n\}$ converges d_0-pointwise to T_0, there is a natural number N such that

$$|d_n(x, y) - d_0(x, y)| < \frac{1-\beta}{4}\varepsilon \quad \text{for all } x, y \in X,$$

$$d_0(u_{n_i}, u_0) < \frac{1-\beta}{4}\varepsilon \quad \text{and} \quad d_0\big(T_n(u_0), T_0(u_0)\big) < \frac{\varepsilon}{2},$$

whenever $n \geqslant N$ and $i \geqslant N$.

Now, taking $i_0 \geqslant N$, we have

$$d_0(u_0, T_0(u_0)) \leqslant d_0(u_0, u_{n_{i_0}}) + d_0(u_{n_{i_0}}, T_0(u_0)) \tag{2}$$

$$< \frac{1-\beta}{4}\varepsilon + d_0(T_{n_{i_0}}(u_{n_{i_0}}), T_0(u_0)).$$

Again,

$$d_0(T_{n_{i_0}}(u_{n_{i_0}}), T_0(u_0)) \leqslant d_0(T_{n_{i_0}}(u_{n_{i_0}}), T_{n_{i_0}}(u_0)) + d_0(T_{n_{i_0}}(u_0), T_0(u_0)) \tag{3}$$

$$< d_{n_{i_0}}(T_{n_{i_0}}(u_{n_{i_0}}), T_{n_{i_0}}(u_0)) + \frac{1-\beta}{4}\varepsilon + \frac{\varepsilon}{2}$$

$$\leqslant \beta\left[d_{n_{i_0}}(u_{n_{i_0}}, T_{n_{i_0}}(u_{n_{i_0}})) + d_{n_{i_0}}(u_0, T_{n_{i_0}}(u_0)\right) + \frac{1-\beta}{4}\varepsilon + \frac{\varepsilon}{2}$$

$$= \beta d_{n_{i_0}}(u_0, T_{n_{i_0}}(u_0)) + \frac{1-\beta}{4}\varepsilon + \frac{\varepsilon}{2}.$$

Also,

$$d_{n_{i_0}}(u_0, T_{n_{i_0}}(u_0)) \leqslant d_{n_{i_0}}(u_0, u_{n_{i_0}}) + d_{n_{i_0}}(u_{n_{i_0}}, T_{n_{i_0}}(u_0))$$

$$< d_0(u_0, u_{n_{i_0}}) + \frac{1-\beta}{4}\varepsilon + d_{n_{i_0}}(T_{n_{i_0}}(u_{n_{i_0}}), T_{n_{i_0}}(u_0))$$

$$< \frac{1-\beta}{2}\varepsilon + \beta\left[d_{n_{i_0}}(u_{n_{i_0}}, T_{n_{i_0}}(u_{n_{i_0}})) + d_{n_{i_0}}(u_0, T_{n_{i_0}}(u_0))\right]$$

$$= \frac{1-\beta}{2}\varepsilon + \beta d_{n_{i_0}}(u_0, T_{n_{i_0}}(u_0)).$$

Therefore,

$$d_{n_{i_0}}(u_0, T_{n_{i_0}}(u_0)) < \frac{\varepsilon}{2}. \tag{4}$$

From (1), (2) and (3) we have

$$d_0(u_0, T_0(u_0)) < \frac{1-\beta}{4}\varepsilon + \frac{\beta}{2}\varepsilon + \frac{1-\beta}{4}\varepsilon + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $u_0 = T_0(u_0)$, i. e., u_0 is a fixed point of T_0.

Note that in the proof of Theorem 3.4 the uniform convergence of $\{d_n\}$ to d_0 has not been used in its full strength. In fact, the uniform convergence can be weakened to a quasi-uniform convergence (for the definition, see [1], p. 139).

THEOREM 3.5. *Suppose (X, d) is a metric space and $T_i: X \to X$ is a map with a fixed point u_i such that*

$$d(T_i(x), T_i(y)) \leqslant \beta [d(x, T_i(x)) + d(y, T_i(y))],$$

where $0 < \beta < 1$ and $i = 1, 2, \ldots$ If $\{T_i\}$ converges pointwise to T_0 that maps X into itself, and if u_0 is a limit point of $\{u_i\}$, then u_0 is a fixed point of T_0.

This is a corollary of Theorem 3.4 if we take $d_n = d$ for all n.

Finally, Example 3.1 shows that Theorems 3.2 and 3.4 do not hold if the uniform convergence of $\{d_n\}$ to d_0 is replaced by pointwise convergence. The example is a modification of one constructed by Nadler and Fraser [3].

Example 3.1. Let $X = \{(2^{-i}, 2^{-j}) \mid i, j = 0, 1, 2, \ldots, \infty\}$ with the convention that $2^{-\infty} = 0$. If $x = (2^{-k}, 2^{-l})$ and $y = (2^{-m}, 2^{-p})$, then, for each integer $n > 0$, let

$$d_n(x, y) = d_n(y, x) = \begin{cases} |2^{-k} - 2^{-m}| & \text{if } l = p = n, \\ 4 - 2^{-k} + 2^{-p} & \text{if } l = n, p \neq n \text{ and } m = 0, \\ 10 & \text{if } l = n, p \neq n \text{ and } m \neq 0, \\ |2^{-l} - 2^{-p}| & \text{if } l \neq n,\ p \neq n \text{ and } m = k = 0, \\ 10 & \text{if } l \neq n, p \neq n, m = 0 \text{ and } k \neq 0, \\ |2^{-k} - 2^{-m}| + |2^{-l} - 2^{-p}| & \text{if } l \neq n, p \neq n, m \neq 0 \text{ and } k \neq 0, \end{cases}$$

and

$$d_0(x, y) = d_0(y, x) = \begin{cases} 10 & \text{if } k = 0 \text{ and } m \neq 0, \\ |2^{-k} - 2^{-m}| + |2^{-l} - 2^{-p}| & \text{if } k \neq 0 \text{ and } m \neq 0, \\ |2^{-l} - 2^{-p}| & \text{if } k = 0 \text{ and } m = 0. \end{cases}$$

It is easy to verify that, for each integer $n \geqslant 0$, d_n is a metric on X and $\{d_n\}$ converges pointwise to d_0.

For each $n > 0$ define $T_n: X \to X$ by

$$T_n(2^{-i}, 2^{-j}) = \begin{cases} (2^{-(i+4)}, 2^{-n}) & \text{if } j = n, \\ (1, 2^{-n}) & \text{if } j \neq n \text{ and } i = 0, \\ (1, 0) & \text{if } j \neq n \text{ and } i \neq 0. \end{cases}$$

Let $T_0: X \to X$ be defined by $T_0(x) = (1, 0)$ for all $x \in X$. Now we verify that

(i) for each $n > 0$, T_n satisfies

$$d_n(T_n(x), T_n(y)) \leqslant \frac{2}{5} [d_n(x, T_n(x)) + d_n(y, T_n(y))] \quad \text{for all } x, y \in X,$$

(ii) $\{T_n\}$ converges d_0-pointwise to T_0,

(iii) T_n has the fixed point

$$u_n = \begin{cases} (0, 2^{-n}) & \text{if } n > 0, \\ (1, 0) & \text{if } n = 0, \end{cases}$$

(iv) $\{u_n\}$ d_0-converges to $(0, 0)$ and not to the fixed point $(1, 0)$ of T_0.

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