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ON THE GREATEST CONGRUENCE RELATION CONTAINED IN AN EQUIVALENCE RELATION AND ITS APPLICATIONS TO THE ALGEBRAIC THEORY OF MACHINES

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In Section 1 of this paper we give a construction of the greatest congruence relation of an arbitrary abstract algebra A that is contained in an equivalence relation E on A (1) and, moreover, we give a construction of a drawing universal object in the category of $\mathrm{mod}\, E$ epimorphisms of the algebra A.

In Section 2 we consider a Moore-type sequential machine of any quasi-algebra A and thus we obtain a form of the subquasi-algebra of A generated by a set X and a concept of X-computation in A.

In Section 3 applications of the considerations of Sections 1 and 2 to the algebraic theory of machines are contained. Moreover, we consider an algebraic theory of $\mathfrak{K}$-machines, where $\mathfrak{K}$ is an arbitrary equationally definable class of algebras, which is a generalization of the usual one. The usual algebraic machines are the $\mathfrak{K}$-machines, where $\mathfrak{K}$ is the class of all semigroups. Finally, we formulate some problems related to the series-parallel composition of $\mathfrak{K}$-machines.

1. A category of $\mathrm{mod}\, E$ epimorphisms of an abstract algebra. We shall consider algebras and quasi-algebras of type G in the sense of paper [6]. Let $G = \{g, \ldots\}$ be any set of operator symbols. By $m(g)$, where $g \in G$, will be denoted the *rank of the operator symbol* g, i.e. the number for which g is m-ary. Any system $\boldsymbol{A} = \langle A, (g_{\boldsymbol{A}}, g \in G)\rangle$, where A is a set and $g_{\boldsymbol{A}}$ is an $m(g)$-ary partial operation in A (i.e. $g_{\boldsymbol{A}}$ is a mapping of a subset of $A^{m(g)}$ into A) for all $g \in G$, is called a *quasi-algebra of type* G. If, moreover, in the system $\boldsymbol{A}$, a $g_{\boldsymbol{A}}$ is an operation in A (i.e. $g_{\boldsymbol{A}}$ is a mapping of the whole set $A^{m(g)}$ into A) for $g \in G$, then $\boldsymbol{A}$ is said to be an *algebra of type* G.

If E is a binary relation on a set A, and E holds between elements a and b, then write aEb or $\langle a, b\rangle \in E$ or $a \equiv b\ (\mathrm{mod}\, E)$.

(1) In the sequel, the set of any quasi-algebra (or algebra) $\boldsymbol{A}$ will be denoted by A.

Let A be an arbitrary algebra of type G. The set of all algebraical operations over algebra A will be denoted by (A), and the set of all m-ary algebraical operations over A will be denoted by $(A)_m$ (cf. [3]).

Definition 1. Let A be an arbitrary algebra of type G and let E be an arbitrary equivalence relation on A. Then, gcE is said to be the *binary relation on* A such that, for all $x, y \in A$, we have $x \equiv y \pmod{gcE}$ if and only if, for all m, for all $f \in (A)_m$ and for all $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m \in A$, the following condition holds:

$$f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_m) \equiv f(a_1, \ldots, a_{i-1}, y, a_{i+1}, \ldots, a_m) \pmod{E}.$$

THEOREM 1. *For every algebra A of type G and for every equivalence relation E on A, the relation gcE is the greatest congruence relation of the algebra A that is contained in E.*

Proof. Obviously, gcE is an equivalence relation on A and if $x \equiv y \pmod{gcE}$, then, putting in Definition 1 $f = I_1^1 = 1_A \in (A)_1$, we obtain $I_1^1(x) \equiv I_1^1(y) \pmod{E}$, i.e. $x \equiv y \pmod{E}$; thus $gcE \subseteq E$.

Now, we shall prove that gcE is a congruence relation of the algebra A. Let $g = g_A$ be any fundamental operation of A and let g be $(m(g))p$-ary. Assume $x_j \equiv y_j \pmod{gcE}$ for $j = 1, \ldots, p$. Let $f \in (A)_m$ and let $1 \leqslant i \leqslant m$. Moreover, let $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m \in A$. We put $m_1 = m+p-1$ and

$$(*) \qquad f_1 = f\big(I_1^{m_1}, \ldots, I_{i-1}^{m_1}, g(I_i^{m_1}, \ldots, I_{i+p-1}^{m_1}), I_{i+p}^{m_1}, \ldots, I_m^{m_1}\big),$$

where $I_i^{m_1}(x_1, \ldots, x_{m_1}) = x_i$ are the trivial m_1-ary algebraical operations over A. By the repeated application of Definition 1 we get, firstly,

$$f_1(a_1, \ldots, a_{i-1}, x_1, x_2, \ldots, x_p, a_{i+1}, \ldots, a_m)$$
$$\equiv f_1(a_1, \ldots, a_{i-1}, y_1, x_2, \ldots, x_p, a_{i+1}, \ldots, a_m) \pmod{E}$$

since $x_1 \equiv y_1 \pmod{gcE}$; secondly,

$$f_1(a_1, \ldots, a_{i-1}, y_1, x_2, \ldots, x_p, a_{i+1}, \ldots, a_m)$$
$$\equiv f_1(a_1, \ldots, a_{i-1}, y_1, y_2, x_3, \ldots, x_p, a_{i+1}, \ldots, a_m) \pmod{E}$$

since $x_2 \equiv y_2 \pmod{gcE}$; and continuing this inference we finally obtain

$$f_1(a_1, \ldots, a_{i-1}, x_1, x_2, \ldots, x_p, a_{i+1}, \ldots, a_m)$$
$$\equiv f_1(a_1, \ldots, a_{i-1}, y_1, y_2, \ldots, y_p, a_{i+1}, \ldots, a_m) \pmod{E}.$$

Hence, by $(*)$, we have

$$f\big(a_1, \ldots, a_{i-1}, g(x_1, \ldots, x_p), a_{i+1}, \ldots, a_m\big)$$
$$\equiv f\big(a_1, \ldots, a_{i-1}, g(y_1, \ldots, y_p), a_{i+1}, \ldots, a_m\big) \pmod{E},$$

and thus, by Definition 1, we obtain

$$g(x_1, \ldots, x_p) \equiv g(y_1, \ldots, y_p) \ (\mathrm{mod}\, gcE)$$

and, therefore, gcE is a congruence relation of the algebra $\boldsymbol{A}$ that is contained in E.

Let $E_1 \subseteq E$ be any congruence relation of the algebra $\boldsymbol{A}$ and let $x \equiv y \ (\mathrm{mod}\, E_1)$. Then, for all m and for all f in $(\boldsymbol{A})_m$, f preserves E_1 and thus

$$f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_m) \equiv f(a_1, \ldots, a_{i-1}, y, a_{i+1}, \ldots, a_m) \ (\mathrm{mod}\, E_1),$$

but $E_1 \subseteq E$, therefore,

$$f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_m) \equiv f(a_1, \ldots, a_{i-1}, y, a_{i+1}, \ldots, a_m) \ (\mathrm{mod}\, E)$$
$$\text{for all } a_1, \ldots, a_{i-1}, \ldots, a_m \text{ in } A.$$

Hence, by Definition 1, $x \equiv y \ (\mathrm{mod}\, gcE)$ and thus $E_1 \subseteq gcE$, i.e. gcE is the greatest congruence relation of the algebra $\boldsymbol{A}$ that is contained in E. This completes our proof of Theorem 1.

From Theorem 1 we immediately obtain

(1.1) *If E is a congruence relation of an algebra $\boldsymbol{A}$, then $gcE = E$.*

Let h: $A \to B$ be any mapping. Then the natural equivalence relation E_h on A induced by h ($a_1 E_h a_2$ iff $h(a_1) = h(a_2)$) will be also denoted by $\mathrm{mod}\, h$. Hence, we have $a_1 \equiv a_2 \ (\mathrm{mod}\, h)$ iff $a_1 \equiv a_2 \ (\mathrm{mod}\, E_h)$.

Now, assume that h: $\boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism (or epimorphism) of an algebra $\boldsymbol{A}$ of type G into an algebra $\boldsymbol{B}$ of the same type G and let E be an equivalence relation on A. Then h is said to be $\mathrm{mod}\, E$ *homomorphism* (or $\mathrm{mod}\, E$ *epimorphism*) *of the algebra* $\boldsymbol{A}$ provided $\mathrm{mod}\, h \subseteq E$. Obviously, $\mathrm{mod}\, h$ is a congruence of $\boldsymbol{A}$ induced by the homomorphism h. Hence, by Theorem 1, we obtain

(1.2) *If h: $\boldsymbol{A} \to \boldsymbol{B}$ is a* $\mathrm{mod}\, E$ *homomorphism of an algebra $\boldsymbol{A}$ of type G, then* $\mathrm{mod}\, h \subseteq gcE$.

Now we formulate

Definition 2. Let $\boldsymbol{A}$ be an algebra of type G and let E be an equivalence relation on $\boldsymbol{A}$. Then the quotient algebra $\boldsymbol{A}^E = \boldsymbol{A}/gcE$ is called the *E-algebra of the algebra $\boldsymbol{A}$*. Moreover, the natural homomorphism, that is the $\mathrm{mod}\, E$ epimorphism of $\boldsymbol{A}$ of the form λ^E: $A \to A^E$ with $\lambda^E(a) = a/gcE$, is said to be the *canonical* $\mathrm{mod}\, E$ *epimorphism of $\boldsymbol{A}$*.

Let $\mathfrak{K}$ be an arbitrary class of algebras of type G closed under the homomorphic images and assume $\boldsymbol{A} \in \mathfrak{K}$.

Define a category of $\mathfrak{K}$-epimorphisms of the algebra $\boldsymbol{A}$ as follows. The objects of this category are $\mathfrak{K}$-epimorphisms of $\boldsymbol{A}$, that is the epimorphisms of the form h: $\boldsymbol{A} \to \boldsymbol{B}$ with $\boldsymbol{B} \in \mathfrak{K}$.

Let $h: \boldsymbol{A} \to \boldsymbol{B}$ and $h_1: \boldsymbol{A} \to \boldsymbol{C}$ be two objects, i.e. two $\mathfrak{K}$-epimorphisms of the algebra $\boldsymbol{A}$. Then the morphisms of the first object into the second one are the epimorphisms $\gamma: \boldsymbol{B} \to \boldsymbol{C}$ such that the diagram

$$\begin{tikzcd} \boldsymbol{A} \arrow[r, "h"] \arrow[dr, "h_1"'] & \boldsymbol{B} \arrow[d, "\gamma"] \\ & \boldsymbol{C} \end{tikzcd}$$

is commutative.

The category of $\mathrm{mod}\,E$ $\mathfrak{K}$-epimorphisms of the algebra $\boldsymbol{A}$ is the subcategory of the one defined above, determined by the objects being $\mathrm{mod}\,E$ epimorphisms of $\boldsymbol{A}$ and the same morphisms. The next theorem determines a universal drawing object of this category.

THEOREM 2. *Let $\boldsymbol{A}$ be an arbitrary algebra of type G and let E be an arbitrary equivalence relation on A. Then the E-algebra $\boldsymbol{A}^E$ of the algebra $\boldsymbol{A}$ is the unique minimal* $\mathrm{mod}\,E$ *homomorphic image of $\boldsymbol{A}$, i.e., for each* $\mathrm{mod}\,E$ *epimorphism $h: \boldsymbol{A} \to \boldsymbol{B}$ of $\boldsymbol{A}$, there exists exactly one epimorphism $\gamma: \boldsymbol{B} \to \boldsymbol{A}^E$ such that the diagram*

$$\begin{tikzcd} \boldsymbol{A} \arrow[r, "h"] \arrow[dr, "\lambda^E"'] & \boldsymbol{B} \arrow[d, "\gamma"] \\ & \boldsymbol{A}^E \end{tikzcd}$$

is commutative.

In other words, the canonical $\mathrm{mod}\,E$ epimorphism $\lambda^E: \boldsymbol{A} \to \boldsymbol{A}^E$ of $\boldsymbol{A}$ is a universal drawing object in the category of all $\mathrm{mod}\,E$ $\mathfrak{K}$-epimorphisms of $\boldsymbol{A}$ for each class $\mathfrak{K}$ of algebras of type G containing $\boldsymbol{A}$ and closed with respect to homomorphic images.

Proof. Since $\boldsymbol{A} \in \mathfrak{K}$, therefore $\boldsymbol{A}^E \in \mathfrak{K}$. Let $h: \boldsymbol{A} \to \boldsymbol{B}$ be any $\mathrm{mod}\,E$ $\mathfrak{K}$-epimorphism of $\boldsymbol{A}$. Putting $\gamma\big(h(a)\big) = a/gcE$, we obtain a well defined mapping because if $h(a_1) = h(a_2)$, i.e. if $a_1 \equiv a_2\,(\mathrm{mod}\,h)$, then, by (1.2), $a_1 \equiv a_2\,(\mathrm{mod}\,gcE)$. Obviously, γ is an epimorphism and $\lambda^E = \gamma h$. This completes our proof of Theorem 2.

Consider a concept of division for algebras.

Definition 3. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be any algebras of type G. We say that $\boldsymbol{A}$ *divides* $\boldsymbol{B}$ (write $\boldsymbol{A}|\boldsymbol{B}$) if there exists an epimorphism $\gamma: \boldsymbol{B}_1 \to \boldsymbol{A}$ such that $\boldsymbol{B}_1$ is a subalgebra of $\boldsymbol{B}$.

Let us notice that the algebra division is a reflexive and transitive relation and, moreover, for finite algebras, it is also an antisymmetric relation (i.e. $\boldsymbol{A}|\boldsymbol{B}$ and $\boldsymbol{B}|\boldsymbol{A}$ and $|A|, |B| < \aleph_0$ implies $\boldsymbol{A}$ is isomorphic to $\boldsymbol{B}$).

By Theorem 2 we have

(1.3) *If h: $\boldsymbol{A}\to\boldsymbol{B}$ is a* mod$E$ *homomorphism of an algebra* $\boldsymbol{A}$ *of type* G, *then* $\boldsymbol{A}^E|\boldsymbol{B}$, *where* $\boldsymbol{A}^E$ *is the* E*-algebra of* $\boldsymbol{A}$.

The next proposition can be easily proved.

(1.4) *If* E_1 *and* E_2 *are any equivalence relations on any algebras* $\boldsymbol{A}_1$ *and* $\boldsymbol{A}_2$ *of type* G *and* $E = E_1\times E_2$ *is the direct product of* E_1 *and* E_2, *then* E *is an equivalence relation on the direct product* $\boldsymbol{A}_1\times\boldsymbol{A}_2$ *of algebras* $\boldsymbol{A}_1$ *and* $\boldsymbol{A}_2$ *and* $gcE = gcE_1\times gcE_2$ *and, therefore,* $\boldsymbol{A}^E$ *is isomorphic to the direct product* $\boldsymbol{A}^{E_1}\times\boldsymbol{A}^{E_2}$ *of* $\boldsymbol{A}^{E_1}$ *and* $\boldsymbol{A}^{E_2}$.

2. A Moore-type sequential machine of a quasi-algebra. A *Moore-type sequential machine* is any quintuple

$$\boldsymbol{M} = \langle S, I, O, \delta, \lambda\rangle,$$

where

(i) S is a non-empty set of states,
(ii) I is a non-empty set of inputs,
(iii) O is a non-empty set of outputs,
(iv) δ: $S\times I\to S$ is called the *transition* (or *next state*) *function*, and
(v) λ: $S\to O$ is called the *output function* [4].

Let $\boldsymbol{M}$ be any Moore-type sequential machine. Define ΣI as the free semigroup freely generated by I. ΣI is the set of all input strings. Define $(\Sigma I)^1$ as the free semigroup with identity freely generated by I. Notice $(\Sigma I)^1 = \Sigma I\cup(\Lambda)$, where Λ is the empty string. We then extend the applicability of δ so that it maps $S\times(\Sigma I)^1$ into S by the repeated application of the equalities

$$\delta(s, \Lambda) = s \quad\text{ and }\quad \delta(s, i_1 i_2) = \delta[\delta(s, i_1), i_2].$$

Define $\bar{\lambda}$: $S\times(\Sigma I)^1\to O$ by $\bar{\lambda}(s, u) = \lambda(\delta(s, u))$ for all s in S and all u in $(\Sigma I)^1$. We may associate with each state s in $\boldsymbol{M}$ the way it produces an output for each input string (the input-output behaviour). This is expressed by the function

(a) $$f\colon \Sigma I\to O, \quad \text{where } f(u) = \bar{\lambda}(s, u) \text{ for each } u\in\Sigma I.$$

Function (a) is also called the *algebraic machine* corresponding to the sequential machine $\boldsymbol{M}$ starting in the state s. The algebraic machines will be considered in the next section.

Let $\boldsymbol{A}$ be an arbitrary quasi-algebra of type G. Define

$$\boldsymbol{M}_A = \langle S_A, I_A, O_A, \delta_A, \lambda_A\rangle$$

by the following:

(1) $S_A = \{s\in A^\omega\colon |s|<\infty\}$, where $s = (s_1, \ldots, s_n, \ldots)$ with $s_n\in A$ for $n = 1, 2, \ldots$, and $|s| = k$ is the least natural number such that $s_m = s_k$ for each $m\geqslant k$;

(2) $I_A = \{\langle g_A, \varphi\rangle$: g_A is the fundamental operation of A induced by $g \in G$ and φ is any permutation of the numbers $1, 2, \ldots, m(g)$, where $m(g)$ is the rank of the operator symbol $g\}$;

(3) $O_A = A$;

(4) for $s = (s_1, \ldots, s_n, \ldots) \in A^{\omega}$, $|s| < \infty$, define $\delta_A(s, \langle g_A, \varphi\rangle) = s' = (s_1', \ldots, s_n', \ldots)$ by

$$s_1' = \begin{cases} g_A\varphi(s_1, \ldots, s_{m(g)}) & \text{if this element is defined (the first case),} \\ s_{m(g)+1} & \text{in the opposite case (the second case),} \end{cases}$$

$$s_j' = \begin{cases} s_{m(g)+j-1} & \text{in the first case} \\ s_{m(g)+j} & \text{in the second case} \end{cases} \quad \text{for } j = 2, 3, \ldots;$$

(5) $\lambda_A(s) = s_1$, where $s = (s_1, \ldots, s_n, \ldots)$ in S_A.

M_A is a Moore-type sequential machine which will be called the *sequential machine of the quasi-algebra* A. If g_A is the $m(g)$-ary fundamental operation of A and φ is any permutation of numbers $1, \ldots, m(g)$, then

$$g_A\varphi(x_1, \ldots, x_{m(g)}) = g_A(x_{\varphi(1)}, \ldots, x_{\varphi(m(g))})$$

(cf. (4) in the above-mentioned definition).

Let X be a subset of a quasi-algebra A of type G and let $C_A(X)$ be the subquasi-algebra of A generated by X. Let ω be the dimension of G, i.e. ω is the least initial number τ such that, for all $g \in G$, τ is not cofinal with any number $a \leqslant m(g)$, and $\tau > m(g)$ (2). Then

$$C_A(X) = \bigcup_{\sigma<\omega} X_\sigma,$$

where X_σ, $\sigma < \omega$, are the *Borel classes of the set* X *in the quasi-algebra* A, i.e.

$$X_0 = X \quad \text{and} \quad X_\tau = \bigcup_{\sigma<\tau} X_\sigma \cup \bigcup_{g\in G} g_A\Big(\bigcup_{\sigma<\tau} X_\sigma\Big),$$

where $g_A(B)$ denotes the set of all elements of A which are the values of the partial operation g_A for elements belonging to $B \subseteq A$.

The least number σ for which an element a belongs to X_σ is said to be the *X-Borel class of a in the quasi-algebra* A.

The elements of $C_A(X)$ can be obtained from X by the input-output behaviour of the sequential machine M_A. This is expressed in the following

(2) In this paper we consider only algebras and quasi-algebras with finitary operations and partial operations. Therefore, $\omega = \omega_0$. Our considerations can be easily generalized for arbitrary quasi-algebras and algebras.

THEOREM 3. $C_A(X) = \{\bar{\lambda}_A(s, u): s \in X^\omega, |s| < \infty, u \in (\Sigma I)^1\}$.

This follows from the definition of M_A. Moreover, let us observe that

(2.1) *The X-Borel class of any element $a \in C_A(X)$ is the least number k such that $a = \bar{\lambda}_A(s, u)$, where $s \in X^\omega$ and k is the length of an input string u of the machine M_A.*

Let us consider a concept of an X-computation in a quasi-algebra A of type G, where X is a subset of A.

Definition 4. A pair $\langle u, a\rangle$ is said to be an *X-computation in the quasi-algebra A of type G* if

(1′) $u = \langle g_{1A}, \varphi_1\rangle \dots \langle g_{pA}, \varphi_p\rangle$ is an input string of M_A;

(2′) $a = (a_1, \dots, a_n)$ is a finite sequence of elements of A which is identical to a sequence obtained as follows:

(a_1) consider a state $s = (s_1, \dots, s_n, \dots)$ of M_A with $s \in X^\omega$;

(a_2) define a finite sequence $s^{(j)}$ $(j = 0, 1, \dots, p)$ of states of M_A by

$$s^{(0)} = s \quad \text{and} \quad s^{(j)} = \delta_A(s^{(j-1)}, \langle g_{jA}, \varphi_j\rangle) \quad \text{for } j = 1, \dots, p;$$

let $s^{(j)} = (s_1^{(j)}, s_2^{(j)}, \dots)$ for $j = 1, \dots, p$;

(a_3) the sequence a is identical to the sequence of the form

$$(s_1, \dots, s_{m(g_1)}, s_1^{(1)}, \dots, s_{m(g_2)}^{(1)}, \dots, s_1^{(p-1)}, \dots, s_{m(g_{p-1})}^{(p-1)}, s_1^{(p)}).$$

The X-computation $\langle u, a\rangle$ is said to be *proper* if the length p of the input string u is the X-Borel class in A of the last element a_n of the sequence a.

If $\langle u, a\rangle$ is an X-computation in a quasi-algebra A of type G, then it is also called an *X-computation of the last element a_n of the sequence a*, and $a_n = \bar{\lambda}_A(s, u)$, where $s \in X^\omega$ is the state from (a_1). Hence, by Theorem 3, we obtain the following proposition:

(2.2) *An element $a \in A$ has an X-computation in A if and only if it belongs to $C_A(X)$. If an element $a \in A$ has an X-computation in A, then it has a proper one.*

Let us consider two quasi-algebras A and B of the same type G and the corresponding sequential machines M_A and M_B. Let

$$u = \langle g_{1A}, \varphi_1\rangle \dots \langle g_{pA}, \varphi_p\rangle$$

be any input string of M_A. Define the corresponding input string u_B of M_B by

$$u_B = \langle g_{1B}, \varphi_1\rangle \dots \langle g_{pB}, \varphi_p\rangle.$$

Let $h: A \to B$ be a mapping and let $s = (s_1, s_2, \dots)$ be a state of M_A. Define the h-image of s by $h(s) = (h(s_1), h(s_2), \dots)$. Obviously, $h(s)$ is a state of M_B.

It is easy to prove the following proposition:

(2.3) *If h is a homomorphism of a quasi-algebra $\boldsymbol{A}$ into a quasi-algebra $\boldsymbol{B}$ of the same type G, then*

(a_1') *h is a homomorphism of $\boldsymbol{M}_{\boldsymbol{A}}$ into $\boldsymbol{M}_{\boldsymbol{B}}$, i.e. $h(\delta_{\boldsymbol{A}}(s,u)) = \delta_{\boldsymbol{B}}(h(s), u_{\boldsymbol{B}})$, and $h(\bar{\lambda}_{\boldsymbol{A}}(s,u)) = \bar{\lambda}_{\boldsymbol{B}}(h(s), u_{\boldsymbol{B}})$ for all s and u;*

(a_2') *if $\langle u, a\rangle$ is an X-computation of an element a in $\boldsymbol{A}$, then $\langle u_{\boldsymbol{B}}, h(a)\rangle$ is an $h(X)$-computation of $h(a)$ in $\boldsymbol{B}$.*

Let a set X generate a quasi-algebra $\boldsymbol{A}$ of type G. Associate with each element $a \in \boldsymbol{A}$ a proper X-computation $\langle u^a, a^a\rangle$ of a in $\boldsymbol{A}$. Denote by $\hat{a}^a = (c_1, c_2, \ldots, c_{k-1}, c_k)$ the sequence obtained from a^a by removing all elements of X except for the first one. $\hat{a}^a$ is a finite sequence of elements of A and, therefore, it defines a state

$$s(\hat{a}^a) = (c_1, \ldots, c_{k-1}, c_k, c_k, \ldots) = (c_1', \ldots, c_n', \ldots),$$

where $c_n' = c_n$ for $n < k$ and $c_n' = c_k$ for $n \geqslant k$. The element

$$a^\sigma = \bar{\lambda}_{\boldsymbol{A}}(s(\hat{a}^a), u^a)$$

is called the *σ-image of a related to the X-computation* $\langle u^a, a^a\rangle$. Moreover, if $f\colon \boldsymbol{A}\to\boldsymbol{B}$ is a mapping of $\boldsymbol{A}$ into a quasi-algebra $\boldsymbol{B}$ of type G, define $f^\sigma\colon \boldsymbol{A}\to\boldsymbol{B}$ by

$$f^\sigma(a) = \bar{\lambda}_{\boldsymbol{B}}(f(s(\hat{a}^a)), u^a_{\boldsymbol{B}}).$$

Obviously, $f^\sigma(x) = f(x)$ for $x \in X$. The mapping f^σ is used in Section 3 in defining the series composition of $\mathfrak{K}$-machines. Now, we show that our definition of f^σ is a generalization of the one given in [2] for semigroups. Indeed, let, as in [2], $A = \Sigma X$ be a free semigroup freely generated by X and let B be any semigroup. Every element $a \in A$ has exactly one proper X-computation in A with the identity permutations. Let $a = x_1x_2\ldots x_n$; then such a unique proper X-computation of a in A is the pair $\langle u^a, a^a\rangle$, where

$$u^a = \underbrace{\langle\cdot, 1\rangle\langle\cdot, 1\rangle\ldots\langle\cdot, 1\rangle}_{n-1 \text{ times}},$$

and 1 is the identity permutation of 1, 2, and

$$a^a = (x_1, x_2, x_1x_2, x_3, x_1x_2x_3, \ldots, x_1x_2\ldots x_{n-1}, x_n, x_1x_2\ldots x_n).$$

Moreover,

$$\hat{a}^a = (x_1, x_1x_2, x_1x_2x_3, \ldots, x_1x_2\ldots x_n),$$

and thus

$$s(\hat{a}^a) = (x_1, x_1x_2, \ldots, x_1x_2\ldots x_{n-1}, a, a, a, \ldots).$$

Hence, if $f\colon A\to B$, then

$$f(s(\hat{a}^a)) = (f(x_1), f(x_1x_2), \ldots, f(x_1x_2\ldots x_{n-1}), f(a), f(a), f(a), \ldots)$$

and

$$f^{\sigma}(a) = \bar{\lambda}_B\big(f(s(\hat{a}^a)), u_B^a\big) = f(x_1)f(x_1x_2)\dots f(x_1\dots x_{n-1})f(a)$$

as in [2].

3. $\mathfrak{R}$-machines. In the algebraic theory of machines (cf. [1]) there are arbitrary mappings of the form $f\colon \Sigma X \to Y$, where ΣX is the free semigroup freely generated by a set X, and Y is any set. The *semigroup of a machine* f is the quotient semigroup $f^S = \Sigma X/\equiv_f$, where $\equiv_f$ is the Myhill [5] congruence relation of ΣX such that $x_1 \equiv_f x_2$ if and only if $f(px_1q) = f(px_2q)$ for all p, q in $(\Sigma X)^1$.

The algebraic machine f is interpreted as a behaviour of input to output of an automaton or a sequential machine (cf. [4]).

It is easy to see that $\equiv_f = gcE_f$ and $f^S = (\Sigma X)^{E_f}$, where $E_f = \operatorname{mod} f$ is the natural equivalence induced by f and $(\Sigma X)^{E_f}$ is the E_f-algebra of the algebra ΣX (cf. Definition 2). Generalizing this conception of a usual algebraic machine, we obtain the notion of a $\mathfrak{R}$-machine. Let $\mathfrak{R}$ be an arbitrary class of algebras of type G closed with respect to homomorphic images and having free algebras freely generated by any sets. In the sequel, we shall assume that $\mathfrak{R}$ is an arbitrary equationally definable class of algebras of type G. Let X be a set. Then, by $\mathfrak{R}(X)$ we shall denote the $\mathfrak{R}$-free algebra freely generated by X.

Let us consider three definitions.

Definition 5. Let X and Y be any sets. Then any mapping of the form $f\colon \mathfrak{R}(X)\to Y$ is called a *$\mathfrak{R}$-machine.*

Definition 6. Let $f\colon \mathfrak{R}(X)\to Y$ be an arbitrary $\mathfrak{R}$-machine. Then the *$\mathfrak{R}$-algebra of the $\mathfrak{R}$-machine* f (write $f^{\mathfrak{R}}$) is the E_f-algebra of $\mathfrak{R}(X)$, where $E_f = \operatorname{mod} f$, i.e. $f^{\mathfrak{R}} = \mathfrak{R}(X)^{E_f} = \mathfrak{R}(X)/gcE_f$ (see Section 1). Moreover, the $\operatorname{mod} E_f$ homomorphisms of $\mathfrak{R}(X)$ are said to be $\operatorname{mod} f$ *homomorphisms.*

Definition 7. Let A be any $\mathfrak{R}$-algebra, that is $A\in\mathfrak{R}$. Then the *$\mathfrak{R}$-machine of A* (write A^f) is the unique homomorphism $A^f\colon \mathfrak{R}(A)\to A$ being the extension of the identity mapping $a\mapsto a$ for $a\in A$.

Now we prove the following theorem:

THEOREM 4. *If A is any $\mathfrak{R}$-algebra, A^f is the $\mathfrak{R}$-machine of A and $A^{f\mathfrak{R}}$ is the $\mathfrak{R}$-algebra of the $\mathfrak{R}$-machine A^f, then $A^{f\mathfrak{R}}$ is isomorphic to A, that is $A^{f\mathfrak{R}} \cong A$.*

Proof. By Definition 7, $A^f\colon \mathfrak{R}(A)\to A$ is a $\operatorname{mod} A^f$ epimorphism of $\mathfrak{R}(A)$, and thus $E_{A^f} = gcE_{A^f}$ by (1.1). Hence

$$A \cong \mathfrak{R}(A)/gcE_{A^f} = \mathfrak{R}(A)^{E_{A^f}} = A^{f\mathfrak{R}}.$$

This completes our proof of Theorem 4.

A fundamental extension of a $\mathfrak{R}$-machine is given by the next theorem.

THEOREM 5. *Let* $f\colon \mathfrak{R}(X)\to Y$ *be any* $\mathfrak{R}$*-machine and let* $f^{\mathfrak{R}}$ *be the* $\mathfrak{R}$*-algebra of* f *and, moreover, let* $f^{\mathfrak{R}f}\colon \mathfrak{R}(f^{\mathfrak{R}})\to f^{\mathfrak{R}}$ *be the* $\mathfrak{R}$*-machine of* $f^{\mathfrak{R}}$. *Then* $f = j_f f^{\mathfrak{R}f} h_f^{\ulcorner}$ *(the fundamental extension of* f*), where*

(b_1) $h_f\colon X\to f^{\mathfrak{R}}$ *with* $h_f(x) = x/gcE_f$, *i.e.* $h_f = \lambda^{E_f}|X$;

(b_2) $h_f^{\ulcorner}\colon \mathfrak{R}(X)\to\mathfrak{R}(f^{\mathfrak{R}})$ *is the unique homomorphism (obviously, length-preserving) being an extension of* h_f *considered as a mapping of* X *into* $\mathfrak{R}(f^{\mathfrak{R}})$;

(b_3) $j_f\colon f^{\mathfrak{R}}\to Y$ *with* $j_f(a/gcE_f) = f(a)$ *for all* $a\in\mathfrak{R}(X)$.

Proof. $f^{\mathfrak{R}f}h_f^{\ulcorner}\colon \mathfrak{R}(X)\to f^{\mathfrak{R}}$ is the unique homomorphism being an extension of $h_f\colon X\to f^{\mathfrak{R}}$. But $\lambda^{E_f}\colon \mathfrak{R}(X)\to f^{\mathfrak{R}}$, which is the canonical $\operatorname{mod} E_f$ epimorphism of $\mathfrak{R}(X)$, is also a homomorphism being an extension of h_f, and thus $\lambda^{E_f} = f^{\mathfrak{R}f}h_f^{\ulcorner}$. Therefore, $f = j_f f^{\mathfrak{R}f} h_f^{\ulcorner}$ since $f = j_f\lambda^{E_f}$. This completes the proof of Theorem 5.

Consider a $\mathfrak{R}$-machine division.

Definition 8. Let $f\colon \mathfrak{R}(X)\to Y$ and $g\colon \mathfrak{R}(Z)\to U$ be two $\mathfrak{R}$-machines. We say that f *divides* g ($f|g$) if there is a homomorphism $H\colon \mathfrak{R}(X)\to\mathfrak{R}(Z)$ and a function $h\colon U\to Y$ such that $f = hgH$, that is if the diagram

$$\begin{tikzcd}
\mathfrak{R}(X) \arrow[r, "f"] \arrow[d, "H"'] & Y \\
\mathfrak{R}(Z) \arrow[r, "g"] & U \arrow[u, "h"']
\end{tikzcd}$$

is commutative.

We say f *divides* g *length-preserving* (write $f|g$ (lp)) if $f|g$ with H being a length-preserving homomorphism, i.e., for each $a\in\mathfrak{R}(X)$, the Borel classes of a with respect to X and of $H(a)$ with respect to Z are identical.

Notice that the $\mathfrak{R}$-machine division and the division (lp) are reflexive and transitive relations. The next theorem gives connections between the $\mathfrak{R}$-algebra and the $\mathfrak{R}$-machine division.

THEOREM 6.

(c_1) *Let* f *be any* $\mathfrak{R}$*-machine. Then* $f|f^{\mathfrak{R}f}$ (lp).

(c_2) *Let* $\boldsymbol{A},\boldsymbol{B}\in\mathfrak{R}$. *Then* $\boldsymbol{A}|\boldsymbol{B}$ *implies* $\boldsymbol{A}^f|\boldsymbol{B}^f$ (lp).

(c_3) *Let* f, g *be any* $\mathfrak{R}$*-machines. Then* $f|g$ *implies* $f^{\mathfrak{R}}|g^{\mathfrak{R}}$.

Proof. (c_1) follows from the fundamental extension for $f = j_f f^{\mathfrak{R}f} h_f^{\ulcorner}$ (see Theorem 5).

(c_2) Suppose $\boldsymbol{A}|\boldsymbol{B}$. Let $\boldsymbol{B}'\subseteq\boldsymbol{B}$ be a subalgebra of $\boldsymbol{B}$ and let $\varphi\colon \boldsymbol{B}'\to\boldsymbol{A}$ be an epimorphism. For each $a\in A$, pick a representative $\bar{a}\in\varphi^{-1}(a)$. Define $h_1\colon A\to B$ by $h_1(a) = \bar{a}$. Define $h_2\colon B\to A$ by $h_2(b) = \varphi(b)$ if $b\in B'$ and it is arbitrary if $b\notin B'$. Let $h_1^{\ulcorner}$ be the unique homomorphism of $\mathfrak{R}(A)$ into $\mathfrak{R}(B)$ being an extension of h_1 considered as a mapping of A into $\mathfrak{R}(B)$. $h_1^{\ulcorner}$ is length-preserving. The mapping

$$\boldsymbol{B}^f h_1^{\ulcorner}\colon \mathfrak{R}(A)\to\boldsymbol{B}'\subseteq\boldsymbol{B}$$

is the unique homomorphism of $\mathfrak{R}(A)$ being an extension of $h_1\colon A \to B$. Hence $\varphi \boldsymbol{B}^f h_1^{\Gamma}\colon \mathfrak{R}(A) \to \boldsymbol{A}$ is a homomorphism of $\mathfrak{R}(A)$ being an extension of the identity mapping $a \mapsto a$ for all $a \in A$. Thus, by Definition 7, $\boldsymbol{A}^f = \varphi \boldsymbol{B}^f h_1^{\Gamma}$ and, therefore, by the definition of h_2, we have $\boldsymbol{A}^f = \varphi \boldsymbol{B}^f h_1^{\Gamma} = h_2 \boldsymbol{B}^f h_1^{\Gamma}$. Hence $\boldsymbol{A}^f | \boldsymbol{B}^f$ (lp).

(c_3) Suppose $f|g$. Then, by (c_1), $f|g^{\mathfrak{R}f}$ (since $f|g$ and $g|g^{\mathfrak{R}f}$), and so $f = hg^{\mathfrak{R}f}H$, where H is a homomorphism. Then $g^{\mathfrak{R}f}H\colon \mathfrak{R}(X) \to g^{\mathfrak{R}}$ is the mod$f$ homomorphism of $\mathfrak{R}(X)$ and, by (1.3) with $E = E_f =$ mod$f$, we have $f^{\mathfrak{R}} | g^{\mathfrak{R}}$. This completes our proof of Theorem 6.

We have seen that, given any $\mathfrak{R}$-machine $f\colon \mathfrak{R}(X) \to Y$, there is a canonical $\mathfrak{R}$-algebra $f^{\mathfrak{R}}$ associated with f, $f^{\mathfrak{R}}$ being the unique minimal homomorphic image with respect to mod$f$ homomorphisms. It is natural to ask the following question:

Suppose we take two $\mathfrak{R}$-machines f and g and combine them to make new $\mathfrak{R}$-machines. Then, how are the $\mathfrak{R}$-algebras of the new $\mathfrak{R}$-machines related to $f^{\mathfrak{R}}$ and $g^{\mathfrak{R}}$?

First, if we take a $\mathfrak{R}$-machine $g\colon \mathfrak{R}(Z) \to U$ and code its input and output sets, i.e. define a homomorphism $H\colon \mathfrak{R}(X) \to \mathfrak{R}(Z)$ and a function $h\colon U \to Y$, we obtain the $\mathfrak{R}$-machine $f = hgH$ of the type $f\colon \mathfrak{R}(X) \to Y$. We noticed that in this case $f^{\mathfrak{R}} | g^{\mathfrak{R}}$ since $f|g$ (cf. Theorem 6 (c_3)).

In the theory of usual machines two obvious ways to hook machines together are series and parallel compositions. Consider these conceptions for $\mathfrak{R}$-machines.

First, consider the case of the parallel composition. Let $f\colon \mathfrak{R}(X) \to Y$ and $g\colon \mathfrak{R}(Z) \to U$ be two $\mathfrak{R}$-machines. Define the parallel composition of f and g to be the $\mathfrak{R}$-machine

$$f \times g\colon \mathfrak{R}(X \times Z) \to Y \times U$$

by putting

$$(f \times g)(a) = \big\langle f\big(\Delta_2(a)\big), g\big(\Delta_1(a)\big)\big\rangle \quad \text{for all } a \in \mathfrak{R}(X \times Z),$$

where $\Delta_i = p_i \Delta$, $i = 1, 2$, with Δ being the unique homomorphism of $\mathfrak{R}(X \times Z)$ into the direct product $\mathfrak{R}(X) \times \mathfrak{R}(Z)$, which is an extension of the identity mapping $\langle x, z\rangle \mapsto \langle x, z\rangle$ for all $\langle x, z\rangle \in X \times Z$ and p_i is the i-th projection. The mappings Δ_i are homomorphisms of the form

$$\Delta_2\colon \mathfrak{R}(X \times Z) \to \mathfrak{R}(X) \quad \text{and} \quad \Delta_1\colon \mathfrak{R}(X \times Z) \to \mathfrak{R}(Z).$$

The parallel composition of any finite number of $\mathfrak{R}$-machines is defined similarly. Let $f_1, \ldots, f_n$ be $\mathfrak{R}$-machines of the form $f_i\colon \mathfrak{R}(X_i) \to Y_i$, $i = 1, \ldots, n$. Let $(f_n \times \ldots \times f_1)^{\mathfrak{R}}$ be the $\mathfrak{R}$-algebra of the parallel composition $f_n \times \ldots \times f_1$ of f_i and let $f_n^{\mathfrak{R}} \times \ldots \times f_1^{\mathfrak{R}}$ be the direct product of $\mathfrak{R}$-algebras $f_i^{\mathfrak{R}}$ of a $\mathfrak{R}$-machine f_i. Then we have

THEOREM 7. $(f_n \times \ldots \times f_1)^{\mathfrak{R}} \mid f_n^{\mathfrak{R}} \times \ldots \times f_1^{\mathfrak{R}}$.

Proof. Let $f_i = j_{f_i} f_i^{\mathfrak{K}f} h_{f_i}^{\ulcorner}$ be the fundamental extension for f_i, $i = 1, \ldots, n$. Then the mapping

$$\theta\colon \mathfrak{K}(X_n \times \ldots \times X_1) \to f_n^{\mathfrak{K}} \times \ldots \times f_1^{\mathfrak{K}}$$

such that

$$\theta(a) = \left\langle f_n^{\mathfrak{K}f}\Big(h_{f_n}^{\ulcorner}\big(\Delta_n(a)\big)\Big), \ldots, f_1^{\mathfrak{K}f}\Big(h_{f_1}^{\ulcorner}\big(\Delta_1(a)\big)\Big)\right\rangle \quad \text{for all } a \in \mathfrak{K}(X_n \times \ldots \times X_1),$$

i.e. $\theta = f_n^{\mathfrak{K}f} h_{f_n}^{\ulcorner} \Delta_n \times \ldots \times f_1^{\mathfrak{K}f} h_{f_1}^{\ulcorner} \Delta_1$, is a $\mathrm{mod}(f_n \times \ldots \times f_1)$ homomorphism of $\mathfrak{K}(X_n \times \ldots \times X_1)$ since

$$f_n \times \ldots \times f_1 = (j_{f_n} \times \ldots \times j_{f_1}) \cdot \theta.$$

Thus, by (1.3), we obtain the assertion of Theorem 7.

If $\mathfrak{K} = \Sigma$ is the class of all semigroups, then from the above-given considerations we obtain the well-known theorems for usual machines (cf. [1] and [2]).

Now consider the difficult case of the series composition of $\mathfrak{K}$-machines. Let $f\colon \mathfrak{K}(X) \to Y$ and $g\colon \mathfrak{K}(Z) \to U$ be two $\mathfrak{K}$-machines. Assume that, for every $a \in \mathfrak{K}(X)$, a proper X-computation of a in $\mathfrak{K}(X)$ is fixed. $f\colon \mathfrak{K}(X) \to Y$ may be also considered as a mapping of the form $f\colon \mathfrak{K}(X) \to \mathfrak{K}(Y)$. Define $f^{\sigma}\colon \mathfrak{K}(X) \to \mathfrak{K}(Y)$ as in the final part of Section 2. Define the series composition of the $\mathfrak{K}$-machine $f\colon \mathfrak{K}(X) \to Y$ and the $\mathfrak{K}$-machine $g\colon \mathfrak{K}(Z) \to U$ with the connecting homomorphism $H\colon \mathfrak{K}(Y) \to \mathfrak{K}(Z)$ to be the $\mathfrak{K}$-machine of the form $gHf^{\sigma}\colon \mathfrak{K}(X) \to U$. From the considerations of the final part of Section 2 it follows that the series composition of $\mathfrak{K}$-machines is a generalization of usual machines (cf. [1] and [2]).

We complete our considerations by formulating some problems related to the series-parallel composition of $\mathfrak{K}$-machines.

Problem 1. How is the $\mathfrak{K}$-algebra of the series composition $(gHf^{\sigma})^{\mathfrak{K}}$ related to $f^{\mathfrak{K}}$ and $g^{\mathfrak{K}}$? (**P 874**)

Let $\mathscr{F}$ be a collection of $\mathfrak{K}$-machines. Define an $\mathrm{SP}(\mathscr{F})$, series-parallel closure of $\mathscr{F}$, to be the least family of $\mathfrak{K}$-machines containing $\mathscr{F}$ and closed under the operations of the series and parallel composition and the division, that is

$$\mathrm{SP}(\mathscr{F}) = \bigcup \{\mathrm{SP}_i(\mathscr{F})\colon i = 1, 2, \ldots\},$$

where $\mathrm{SP}_1(\mathscr{F}) = \mathscr{F}$ and $\mathrm{SP}_i(\mathscr{F}) = \{f_2 \times f_1, f_2 h^{\ulcorner} f_1^{\sigma}, h_2 f h_1^{\ulcorner}\colon f_1, f_2$ belong to $\mathrm{SP}_{i-1}(\mathscr{F})$ and h, h_1, h_2 are functions$\}$.

Problem 2. What can be said about the $\mathfrak{K}$-algebras of $\mathfrak{K}$-machines in $\mathrm{SP}(\mathscr{F})$, that is about the class $\mathrm{SP}(\mathscr{F})^{\mathfrak{K}} = \{f^{\mathfrak{K}}\colon f$ is in $\mathrm{SP}(\mathscr{F})\}$? (**P 875**)

Problem 3. Let f be a $\mathfrak{K}$-machine. What are the families $\mathscr{F}$ of $\mathfrak{K}$-machines such that $f \in \mathrm{SP}(\mathscr{F})$? (**P 876**)

Problems 1, 2 and 3 are solved in the case $\mathfrak{K} = \Sigma$, where Σ is the class of all semigroups (cf. [1] and [2]).

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