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INJECTIVITY IN MODEL THEORY

BY

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The aim of this paper is to relate two obvious approaches to the study of an equational class of algebras: the diagrammatic study of the associated category, and the model-theoretic study of the associated theory. Paramount among diagrammatic properties of objects in a category is that of injectivity with respect to natural classes of mappings, and we shall be interested mainly in such properties. Topics considered include the Amalgamation Property, homogeneous universal algebras, equational compactness and purity.

Some of the results here are taken from [1]. The author acknowledges with pleasure the helpful conversations he has had with Dr. G. Sabbagh on certain topics and the motivation derived from a study of [10].

1. Preliminaries. The diagrammatic techniques are borrowed from the *theory of bicategories* (see [12]), but it has been thought best for expository purposes to suppress explicit mention of this (although the knowledgeable reader will have no trouble in reconstructing the general form of the arguments). We shall in fact apply our methods to posets, but in this simple case the modification of the algebraic techniques will be clear. For simplicity all languages will be assumed countable (the general case is not much harder, but somewhat less elegant).

The first definition introduces three key diagrammatic properties of a variety.

Definition 1.1.

(1) Let $\mathscr{C}$ be a variety. $\mathscr{C}$ is said to have the *Amalgamation Property* (AP) if, whenever f and g are injections (i.e. 1-1 homomorphisms) with the same domain, there are injections h and k with $hf = kg$.

(2) $\mathscr{C}$ has the *Congruence Extension Property* (CEP) if, whenever f is an injection and g a surjection (i.e. an onto homomorphism) with domain that of f, there is a surjection h and injection k with $hf = kg$.

(3) *Injections are transferable* in $\mathscr{C}$ if, whenever f is an injection and g a morphism with domain that of f, there is a morphism h and injection k with $hf = kg$.

The Amalgamation Property was introduced in this form by Jónsson in [11], while the choice of name for Property (2) is clear from the following easy lemma:

LEMMA 1.2. *$\mathscr{C}$ has CEP iff whenever B is a $\mathscr{C}$-algebra, A a subalgebra of B, and J a congruence on A, there is a congruence K on B with $K \cap A^2 = J$.*

The following definition and theorem are well known to category-theorists.

Definition 1.3.

(1) A commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

is a *pushout diagram* if whenever h' and k' are morphisms with $h'f = k'g$, there is a unique u with $uh = h'$ and $uk = k'$.

(2) $\mathscr{C}$ is said to *have pushouts* if every diagram $C \leftarrow A \rightarrow B$ can be enlarged to a pushout diagram.

THEOREM 1.4. *If $\mathscr{C}$ is a variety or the category of posets, $\mathscr{C}$ has pushouts.*

Proof. Since $\mathscr{C}$ has coproducts and coequalisers.

Now let X be an abstract class of morphisms. We say that *pushouts transfer X-morphisms* if in any pushout diagram, with the notation of 1.3(1), $f \in X$ implies $k \in X$.

By applying Proposition 1.1(9) of [12] to varieties or posets, the next lemma follows:

LEMMA 1.5 (Kennison). *Pushouts transfer surjections.*

We conclude this section with two simple lemmas.

LEMMA 1.6. *Injections are transferable in $\mathscr{C}$ iff pushouts transfer injections in $\mathscr{C}$.*

Proof. Sufficiency is obvious. Conversely, in the notation of 1.3(1), let f be an injection. Now there is a morphism h' and injection k' with $h'f = k'g$. Hence there is u with $uk = k'$, and so k is an injection.

LEMMA 1.7. *Injections are transferable in $\mathscr{C}$ iff $\mathscr{C}$ has AP and CEP.*

Proof. Necessity follows by 1.6 and 1.5. Conversely, let f be an injection and g a morphism with domain that of f. Let $g = g_0 g_1$, where g_1 is a surjection and g_0 an injection. By CEP let h and k be morphisms with $hf = kg_1$ and k an injection. By AP there are injections u and v with $uk = vg_0$. Then $vg = (uh)f$.

2. The Amalgamation Property and Injectivity. We now show how the well-known concept of *injectivity* can be used to give a general proof of the Amalgamation Property.

Definition 2.1. Let A be an algebra of $\mathscr{C}$, and $u: B \to C$ a morphism of $\mathscr{C}$. A is called *u-injective* if, for each morphism $f: B \to A$, there is $g: C \to A$ with $gu = f$. A is called *injective* if A is u-injective for all injections u of $\mathscr{C}$. Finally, $\mathscr{C}$ is said to have *enough injectives* if, for every algebra A, there is an injection $A \to B$ with B injective.

THEOREM 2.2. *Suppose $\mathscr{C}$ has enough injectives. Then injections are transferable, and so $\mathscr{C}$ has AP and CEP.*

Proof. Let $f: A \to B$ and $g: A \to C$ be morphisms with f an injection. Let $k: C \to D$ be an injection with D injective. Then there is $h: B \to D$ with $hf = kg$.

The following varieties are known to have enough injectives:

(1) $\mathscr{B}$: boolean algebras — as injective means complete (see 33.1 and 35.1 of [19]);

(2) $\mathscr{A}$: abelian groups — as injective means divisible;

(3) distributive lattices — by Corollary 2, p. 104, of [6];

(4) $\mathscr{D}$: distributive e-lattices (i.e., with 0 and 1 as constants) — by modifying the proof of (3), or by [3];

(5) semilattices — by [8];

(6) Stone algebras — by [4].

Hence all these have AP and CEP. In addition, if $\mathscr{P}$ is the category of posets, $\mathscr{P}$ has enough injectives (as injective means complete by [5]), and so the obvious modification of the arguments (1)-(6) shows that $\mathscr{P}$ has AP and CEP.

Note that result (3) solves a problem of [11] (i.e. Remark 2, p. 141).

3. Homogeneous universal algebras. Let $\mathfrak{n}$ be an infinite cardinal. An algebra A is called $\mathfrak{n}$-*universal* if, for every non-trivial algebra B of power $\leqslant \mathfrak{n}$, there is an injection $B \to A$, and $\mathfrak{n}$-*homogeneous-universal* if, in addition, for any two injections $u, v: B \to A$ with $|B| < \mathfrak{n}$, there is an automorphism e of A with $eu = v$. A is called *homogeneous-universal* (HU) if A is $\mathfrak{n}$-HU, where $\mathfrak{n} = |A|$.

$\mathscr{C}$ is said to have the *Joint Embedding Property* (JEP) if, whenever A and B are non-trivial algebras, there are injections $A \to C$ and $B \to C$ for some algebra C.

The next theorem is a well-known result of Jónsson (see [11]).

THEOREM 3.1 (GCH). *Let $\mathscr{C}$ be a variety with AP and JEP. Then, for each regular uncountable cardinal $\mathfrak{n}$, there is an essentially unique HU algebra of power $\mathfrak{n}$. If, in addition, $\mathscr{C}$ is locally finite, there is a denumerable HU algebra.*

Thus, in particular, $\mathscr{D}$ has HU algebras of every regular cardinality, and we shall now characterize these. To do this we need Lemma 3.2 (noting first that $\mathscr{B}$ can be regarded as a full subcategory of $\mathscr{D}$). The notion of *reflection* can be found in [12].

LEMMA 3.2. *$\mathscr{B}$ is reflective in $\mathscr{D}$, the reflection of an injection is an injection, and every reflection morphism is an injection.*

Proof. This was established by Nerode in [15] using topological methods. An algebraic proof is given in [1] (see Corollary 6.4.4).

We call the reflection D^* of D the *enveloping boolean algebra* of D. It is easy to see that if D is finite, so is D^*, and, otherwise, $|D^*| = |D|$. With these remarks we can now establish

THEOREM 3.3. *If B is $\mathfrak{n}$-HU for $\mathscr{B}$, then it is $\mathfrak{n}$-HU for $\mathscr{D}$.*

Proof. Let $D \in \mathscr{D}$ have $1 < |D| \leqslant \mathfrak{n}$. Let $u: D \to D^*$ be the reflection injection for D. Now $|D^*| \leqslant \mathfrak{n}$, and so there is an injection $e: D^* \to B$. Then eu is an injection. Now let $e, e': D \to B$ be injections, where $|D| < \mathfrak{n}$. Then there are injections $f, f': D^* \to B$ with $fu = e$ and $f'u = e'$. Since $|D^*| < \mathfrak{n}$, there is an automorphism h of B with $hf = f'$. Then $he = e'$.

4. Purity. First we introduce the following

Definition 4.1.

(1) Let A be a subalgebra of an algebra B. Then A is said to be a *pure* subalgebra of B if whenever Σ is any finite set of equations over A which is satisfiable in B, then Σ is satisfiable in A. An injection $u: A \to B$ is said to be *pure* if uA is a pure subalgebra of B.

(2) An algebra A is called *absolutely pure* if every injection with domain A is pure (this is called algebraically closed in [10]).

(3) Similarly one can define $\mathfrak{n}$-*pure* injection and *absolutely* $\mathfrak{n}$-*pure* algebra for any cardinal $\mathfrak{n} > \omega$.

Two facts to note about pure injections are (1) the composition of pure injections is a pure injection, and (2) if $A \to B \to C$ is a pure injection, then $A \to B$ is a pure injection. $\mathscr{C}$ always has enough absolutely pure objects, on account of the following lemma of Scott in [18]:

LEMMA 4.2. *For every algebra A there is an injection $A \to B$, where B is absolutely pure and $|B| = \max(\omega, |A|)$.*

However, absolutely pure objects also arise from certain kinds of injectives, as we shall now see.

Definition 4.3. A is called a *local injective* if A is u-injective for each injection $u: B \to C$ with B and C finitely generated.

LEMMA 4.4. *Any local injective is absolutely pure.*

Proof. Let A be a subalgebra of an algebra B, and A be a local injective. Let Σ be a finite system of equations over A in the variables $x_1, \ldots, x_m$, say. Assume Σ is satisfiable in B. Let $a_1, \ldots, a_k$ be the elements of A appearing in Σ. Let A_0 be the subalgebra of A generated by $a_1, \ldots, a_k$, and B_0 — the subalgebra of B generated by $a_1, \ldots, a_k$ and some solution $x_1, \ldots, x_m$ of Σ in B. Now $A_0 \subseteq B_0$ and both are finitely generated. Since $A_0 \subseteq A$, there is a morphism $r\colon B_0 \to A$ with $r(a) = a$ for $a \in A_0$. Then $r(x_1), \ldots, r(x_m)$ is a solution of Σ in A.

One closure property that the absolutely pure algebras always enjoy is given by the next lemma, which is due to G. Sabbagh.

LEMMA 4.5. *Absolutely pure algebras are inductive.*

Proof. Let A be an algebra which is the union of a directed system (A_i) of subalgebras of A with A_i absolutely pure for each i. Let $A \to B$ be an extension of A, and Σ a finite set of equations over A satisfiable in B. Let $a_1, \ldots, a_k$ be the elements of A appearing in Σ. Now there is i with $a_1, \ldots, a_k \in A_i$. As $A_i \to A \to B$ is pure, Σ is satisfiable in A_i, and so in A.

Hence in $\mathscr{B}$ every algebra A is absolutely pure. For A is the directed union of its finite subalgebras, each of which is injective, and so absolutely pure by 4.4.

5. Bounded injectivity. We start with the following

Definition 5.1. An algebra A is $\mathfrak{n}$-*injective* if A is u-injective for each injection $u\colon B \to C$ with $|C| < \mathfrak{n}$.

By successive extension it is easy to establish that if A is $\mathfrak{n}$-injective, it is in fact u-injective for all injections $u\colon B \to C$ with $|B| < \mathfrak{n}$ and $|C| \leqslant \mathfrak{n}$. Clearly, every injective is $\mathfrak{n}$-injective. But a more interesting source of $\mathfrak{n}$-injectives is provided by the next theorem.

THEOREM 5.2. *Let $\mathscr{C}$ have CEP. Then any $\mathfrak{n}$-HU is $\mathfrak{n}$-injective.*

Proof. Let A be $\mathfrak{n}$-HU, $u\colon B \to C$ be an injection with $|C| < \mathfrak{n}$, and $f\colon B \to A$ be a morphism.

(1) Suppose first that f is an injection. Then let $e\colon C \to A$ be an injection (as A is $\mathfrak{n}$-universal). Now $eu, f\colon B \to A$ are injections, and so there is an automorphism v of A with $v(eu) = f$. Then $ve\colon C \to A$ is such that $(ve)u = f$.

(2) In general, let $f = f_0 f_1$, where $f_0\colon D \to A$ is an injection, and $f_1\colon B \to D$ a surjection. By CEP, there is an injection $h\colon D \to E$ and a surjection $k\colon C \to E$ with $ku = hf_1$. Now $|E| \leqslant |C| < \mathfrak{n}$. Hence, by (1), there is $g\colon E \to A$ with $gh = f_0$. Thus $(gk)u = ghf_1 = f$.

Theorem 5.2 for $\mathscr{B}$ is a result of Mączyński [13] and in fact the proof given here is motivated by his proof. The possibility of generalizing the proof in [13] was pointed out to the author by B. Rotman — in addition, G. Sabbagh informs me that he has obtained a similar result independently.

We shall now consider $\mathfrak{n}$-injectives in $\mathscr{P}$ and $\mathscr{B}$. A poset P will be said to have the $\mathfrak{n}$-*Interpolation Property* ($\mathfrak{n}$-IP) if whenever A and B are subsets of P of power less than $\mathfrak{n}$ and $A \leqslant B$ there is $x \in P$ with $A \leqslant x \leqslant B$. Then the following is true.

LEMMA 5.3. *Let* $\mathfrak{n} \geqslant \omega$. *Then, in* (a) $\mathscr{P}$ *or* (b) $\mathscr{B}$, *an object* A *with the* $\mathfrak{n}$-*IP is* $\mathfrak{n}$-*injective.*

Proof. Note first that it is enough to show that A is u-injective for *simple* extensions $u\colon B \to C$ with $|C| < \mathfrak{n}$. Then (a) is obvious and (b) follows from the proof of Theorem 33.1 of [19].

The converse of 5.3 is true in both cases, but for the moment we only prove Part (a). Assume P is $\mathfrak{n}$-injective and that A and B are subsets of P of power less than $\mathfrak{n}$ with $A < B$ (as if $A \cap B \neq \emptyset$ there is x with $A \leqslant x \leqslant B$). Let $C = A \cup B$ with the induced order, and $D = C \cup \{x\}$, where $A < x < B$. Then there is $g\colon D \to P$ with $g(c) = c$ for $c \in C$ (as $|D| < \mathfrak{n}$). Thus $A \leqslant g(x) \leqslant B$ in $\mathscr{P}$.

However, there are many varieties with *no* non-trivial injectives. It is interesting to note that in several of these cases it can in fact be shown that there are not even any non-trivial *local* injectives. As an example we consider the variety $\mathscr{G}$ of groups.

We recall that, in Theorem 2 of [2], Baer proved that $\mathscr{G}$ has no non-trivial injectives.

THEOREM 5.4. $\mathscr{G}$ *has no non-trivial local injectives.*

Proof. Let A be a local injective, $|A| > 1$. Let Z be the infinite cyclic group, $Z*Z$ the free product of Z with itself, and $p\colon Z*Z \to Z$ — the canonical surjection. By a result of [9], there is an injection $e\colon Z*Z \to S$, where S is a finitely generated simple group. Take $x \in A$ with $x \neq 1$ and define $u\colon Z \to A$ by $u(g) = x$ (where g is a generator of Z). If $f\colon S \to A$ is a morphism with $fe = up$, then f is an injection (as S is simple and u has non-trivial range), and so p is an injection. But p is a proper surjection. Hence no such f exists, and so A is not e-injective.

Note that the same type of argument using the diagram $Z \leftarrow Z*Z \to S$ shows that $\mathscr{G}$ does not have CEP. However, $\mathscr{G}$ has AP (see [16]), and so CEP is not a consequence of AP.

Lastly, we show that if injections are transferable, absolutely pure algebras have the following injectivity property:

An algebra A is called a *hyperlocal injective* if A is u-injective for all injections $u\colon B \to C$, where B is finitely generated and C is *finitely presented.*

THEOREM 5.5. *Suppose that injections are transferable. Then any absolutely pure algebra is a hyperlocal injective.*

Proof. Let A be absolutely pure, C a finitely presented algebra, and B a finitely generated subalgebra of C. Let $f\colon B \to A$ be a morphism. Then there is an algebra D such that A is a subalgebra of D and a morphism $g\colon C \to D$ such that $g(b) = f(b)$ for $b \in B$. Let $(S; W)$ be a finite presentation of C. List S as $s_1, \ldots, s_n$. Let $u_1, \ldots, u_k$ be words in S whose values in C form a set of generators of B. Take variables $x_1, \ldots, x_n$. For any word w over S let $w(x)$ be the result of replacing s_i by x_i in w for $1 \leqslant i \leqslant n$. Now consider the following finite system Σ of equations over A:

$$u_i(x) = f(u_i) \quad \text{for } 1 \leqslant i \leqslant k,$$
$$w(x) = w'(x) \quad \text{for each pair } (w, w') \in W.$$

Clearly, Σ is satisfiable in D with $x_i = g(s_i)$, and as A is a pure subalgebra of D, Σ is satisfiable in A. Let $x_i = a_i$ be a solution. Then there is a unique morphism $h\colon C \to A$ given by $h(s_i) = a_i$, and $h(b) = f(b)$ for $b \in B$.

We shall say that a variety $\mathscr{C}$ is *locally finitely presented* if every finitely generated algebra of $\mathscr{C}$ is finitely presented. Then we have the following corollary:

COROLLARY 5.6. *Let $\mathscr{C}$ be locally finitely presented, and suppose that injections are transferable in $\mathscr{C}$. Then an algebra is absolutely pure iff it is a local injective.*

Proof. Necessity by 5.5, and sufficiency by 4.4.

Note that the following varieties are locally finitely presented: boolean algebras, distributive e-lattices, semilattices, and abelian groups. Thus Corollary 5.6 applies to all of these. By the obvious modification of the arguments, it applies to the category of posets as well, and we obtain the result that a poset is absolutely pure iff it has the ω-IP.

For $\mathfrak{n} > \omega$, the situation for absolute $\mathfrak{n}$-purity is more pleasant. For if A is an algebra, A has a natural presentation $(A; D)$, where D is the set of all equations over A holding in A. Thus a trivialisation of the method of 5.5 and a generalization of the method of 4.4 yield the result that in a variety where injections are transferable, an object is $\mathfrak{n}$-injective iff it is absolutely $\mathfrak{n}$-pure.

6. Equational compactness. In this section we elucidate the connection between equational compactness and injectivity. The first definition is taken from Mycielski [14].

Definition 6.1. Let $\mathfrak{n}$ be an uncountable cardinal. An algebra A is *$\mathfrak{n}$-equationally compact* ($\mathfrak{n}$-EC) if whenever Σ is a set of equations over A of power less than $\mathfrak{n}$ and every finite subset of Σ is satisfiable in A, then Σ is satisfiable in A. A is called *equationally compact* if A is $\mathfrak{n}$-equationally compact for all $\mathfrak{n}$.

The correct notion of injectivity to use is the following:

Definition 6.2. A is *$\mathfrak{n}$-pure-injective* if A is u-injective for all pure injections $u\colon B \to C$ with $|C| < \mathfrak{n}$.

LEMMA 6.3. *Let $\mathfrak{n} > \omega$. If A is $\mathfrak{n}$-pure-injective, then A is $\mathfrak{n}$-equationally compact.*

Proof. Let Σ be a set of equations over A of power less than $\mathfrak{n}$ with every finite subset satisfiable in A. Let C be an elementary subalgebra of A containing all the elements of A appearing in Σ and such that $|C| < \mathfrak{n}$, by the Downward Löwenheim-Skolem Theorem. By an application of the Compactness Theorem, there is an elementary extension $A \to A^*$ of A with Σ satisfiable in A^*. Let B be an elementary subalgebra of A^* containing C and some solution (x_i) of Σ in A^* and such that $|B| < \mathfrak{n}$. Thus Σ is satisfiable in B. Now the inclusion $C \to B$ is elementary and $|B| < \mathfrak{n}$. Hence, by hypothesis, there is a morphism $r\colon B \to A$ with $r(c) = c$ for $c \in C$. Then $\big(r(x_i)\big)$ is a solution of Σ in A.

One immediate application of this is a characterization of $\mathfrak{n}$-EC boolean algebras which answers a question of Węglorz (see [20], p. 298).

COROLLARY 6.4. *Let $\mathfrak{n} > \omega$. Then in the variety of boolean algebras, the following properties of an algebra A are equivalent:*

(1) *A has $\mathfrak{n}$-IP;*

(2) *A is $\mathfrak{n}$-injective;*

(3) *A is $\mathfrak{n}$-EC.*

Proof. (1) $\to$ (2) follows from 5.3(b); (2) $\to$ (3) follows from 6.3; and (3) $\to$ (1) is an easy argument (see [20], p. 296).

Now let $\mathfrak{n} > \omega$ be regular, and let B be an $\mathfrak{n}$-saturated model of the theory of atomless boolean algebras (using Theorem 1.7 of Chapter 11 of [7]). Then clearly B has $\mathfrak{n}$-IP. But B has a strictly increasing ω-sequence (a_n), and no such sequence can have a sup. Thus B is not even ω_1-complete.

We now prove a converse to 6.3, and to do this we introduce the notion of the *equation-system* of a diagram. Let $A \xleftarrow{f} B \to C$ be a diagram, where $B \to C$ is an inclusion. Then the equation-system Σ of this diagram is defined as follows. Take variables x_c, $c \in C$, and let Σ consist of the following equations over A:

(1) $x_b = f(b)$ for $b \in B$;

(2) $x_c = \sigma(x_{c(1)}, \ldots, x_{c(n)})$ for every operation symbol σ, of rank n, for instance, and each sequence $c, c(1), \ldots, c(n)$ of elements of C with $c = \sigma\big(c(1), \ldots, c(n)\big)$ in C.

Note that $x_c = g(c)$ for $c \in C$ is a solution of Σ in A iff $g\colon C \to A$ s a morphism extending f.

THEOREM 6.5. *Let $\mathscr{C}$ be a variety, and $A \xleftarrow{f} B \to C$ a diagram in $\mathscr{C}$ with $B \to C$ an inclusion. Then the equation-system Σ of this diagram is finitely satisfiable in A if* (a) *A is a local injective or* (b) *$B \to C$ is pure.*

Proof.

(a) Let Σ' be a finite subset of Σ, $S = \{c \in C\colon x_c \text{ occurs in } \Sigma'\}$, and $R = S \cap B$. Let C' be the subalgebra of C generated by S, B' be the subalgebra of B generated by R, and $f' = f|B'$. Then $B' \subseteq C'$ and both are finitely generated. Thus there is a morphism $g\colon C' \to A$ with $g|B' = f'$. Clearly, $x_c = g(c)$ for $c \in S$ is a solution of Σ' in A.

(b) Let Σ' be a finite subset of Σ, Σ^* consist of those equations in Σ' of type (2) together with $x_b = b$ for each equation $x_b = f(b)$ of type (1) in Σ'. Let $c(1), \ldots, c(k) \in C$ be the indices of the variables occurring in Σ^*. Now Σ^* is satisfiable in C, with $x_{c(i)} = c(i)$, and as B is pure in C, Σ^* is satisfiable in B. Let $x_{c(i)} = b_i$ be a solution of Σ^* in B. Then $x_{c(i)} = f(b_i)$ is a solution of Σ' in A.

This result has three important consequences.

THEOREM 6.6. *Let $\mathfrak{n} > \omega$. Then an algebra A is $\mathfrak{n}$-injective iff it is a local injective and $\mathfrak{n}$-equationally compact.*

Proof. Necessity follows by 6.3, and sufficiency by 6.5(a).

THEOREM 6.7. *Let $\mathfrak{n} > \omega$. Then an algebra A is $\mathfrak{n}$-equationally compact iff it is $\mathfrak{n}$-pure-injective.*

Proof. Necessity follows from 6.5(b), once we note that $|\Sigma| < \mathfrak{n}$ if $|C| < \mathfrak{n}$. Sufficiency follows from 6.3.

LEMMA 6.8. *Assume the hypotheses of Theorem 6.5. Then there is an elementary injection $e\colon A \to A^*$ and a morphism $g\colon C \to A^*$ with $g|B = ef$.*

Proof. By the Compactness Theorem, let $e\colon A \to A^*$ be an elementary injection such that Σ is satisfiable in A^*. Let $x_c = g(c)$ for $c \in C$ be a solution of Σ in A^*. Then $g\colon C \to A^*$ is a morphism, and $g(b) = x_b = e(f(b))$ for $b \in B$.

Hence we can establish

THEOREM 6.9. *If $\mathscr{C}$ has enough local injectives, then injections are transferable in $\mathscr{C}$.*

Proof. Consider the diagram $A \xleftarrow{f} B \xrightarrow{u} C$, where u is an injection. Let $d\colon A \to D$ be an injection with D a local injective. Then there is an injection $e\colon D \to D^*$ and a morphism $g\colon C \to D^*$ such that $gu = e(df) = (ed)f$, by 6.8.

Thus we have found another way of establishing AP and CEP for varieties. However, Theorem 2.2 is not in practice superseded, and it is useful for categories in which the Compactness Theorem is not available (e.g., metric spaces).

Finally, we note that Theorem 6.9 has a strong converse, as follows.

THEOREM 6.10. *Suppose that injections are transferable in $\mathscr{C}$. Then, for each $\mathfrak{n} > \omega$, $\mathscr{C}$ has enough $\mathfrak{n}$-injectives.*

Proof. It suffices to consider regular $\mathfrak{n}$. Let A be a non-trivial algebra. By using the method of the Homogenization Theorem 3.4 in Chapter 10 of [7], i.e., repeated applications of AP, we obtain an extension H of A which has the property of Part (1) of 5.2. By CEP, H is then $\mathfrak{n}$-injective (as in 5.2).

7. Distributive lattices. We use some of the preceding techniques to characterize the absolutely pure and $\mathfrak{n}$-injective objects in the category $\mathscr{D}$ of distributive e-lattices.

Let A be the 4-element algebra $\{0, 1, a, b\}$ with $a \wedge b = 0$ and $a \vee b = 1$, B the subalgebra $\{0, a, 1\}$, and u: $B \to A$ the inclusion. The next result is easy to establish.

LEMMA 7.1. *In $\mathscr{D}$, C is boolean iff C is u-injective.*

We can now characterize the $\mathfrak{n}$-injectives in $\mathscr{D}$. Note that the method which will be used in (2) is essentially due to Balbes (see [3], Lemma 3.1).

THEOREM 7.2. *Let $\mathfrak{n} \geqslant \omega$. Then in $\mathscr{D}$, C is $\mathfrak{n}$-injective iff C is boolean and has $\mathfrak{n}$-IP.*

Proof.

(1) Let C be $\mathfrak{n}$-injective. Then C is u-injective, and so boolean. Since $\mathscr{B}$ is isomorphic to a full subcategory of $\mathscr{D}$, C is $\mathfrak{n}$-injective in $\mathscr{B}$, and so has $\mathfrak{n}$-IP.

(2) Conversely, let e: $D \to E$ be an injection, where $|E| < \mathfrak{n}$, and f: $D \to C$ a morphism, where C is boolean and has $\mathfrak{n}$-IP. Let e^*: $D^* \to E^*$ be the reflection of e, and f^*: $D^* \to C$ the reflection of f. Let i_D: $D \to D^*$ and i_E: $E \to E^*$ be the reflection morphisms. Then $e^* i_D = i_E e$, and $f^* i_D = f$. Since D^*, E^* and C belong to $\mathscr{B}$, and C is $\mathfrak{n}$-injective in $\mathscr{B}$, let h: $E^* \to C$ be such that $he^* = f^*$ (as $|E^*| < \mathfrak{n}$ and e^* is an injection). Then $(hi_E)e = he^* i_D = f^* i_D = f$.

COROLLARY 7.3. *In $\mathscr{D}$, C is absolutely pure iff C is boolean.*

Proof. If C is absolutely pure, C is ω-injective, by 5.6, and so boolean by 7.2. The converse follows from 7.2 and 4.4.

By 7.1 we can now establish the converse of 3.3. For if C is $\mathfrak{n}$-HU in $\mathscr{D}$, it is boolean by 7.1, and so, clearly, $\mathfrak{n}$-HU in $\mathscr{B}$.

8. Existential formulas and orthoinjectivity. In this final section we outline a modification of the previous techniques which will enable us to deal with existential formulas. The basic idea is to restrict all morphisms to be injections.

Definition 8.1. An algebra A is *u-orthoinjective* (where u: $B \to C$ is a morphism) if, for every injection f: $B \to A$, there is an injection g: $C \to A$ with $gu = f$. A is *$\mathfrak{n}$-orthoinjective* if A is u-orthoinjective for all injections u: $B \to C$ with $|C| < \mathfrak{n}$.

Part (1) of 5.2 clearly establishes that any $\mathfrak{n}$-HU is $\mathfrak{n}$-orthoinjective. Now we recall from Section 9.3 of [7] the notion of *existential* formula. By replacing "injective" by "orthoinjective" and "equation" by "existential formula" many of the preceding results have natural modifications. We shall consider the most important one only.

Definition 8.2. An algebra A is *$\mathfrak{n}$-existentially compact* if whenever Σ

is a set of existential formulas over A of power less than $\mathfrak{n}$, and every finite subset of Σ is satisfiable in A, then Σ is satisfiable in A.

THEOREM 8.3. *Any $\mathfrak{n}$-orthoinjective is $\mathfrak{n}$-existentially compact.*

Proof. By modifying 6.3, the notation of which we use. Let A be $\mathfrak{n}$-orthoinjective. Since $C \to A$ is an inclusion, there is an *injection* r: $B \to A$ with $r(c) = c$ for $c \in C$. As Σ is satisfiable in B and existential formulas are preserved under extension, Σ is satisfiable in A.

COROLLARY 8.4. *Let T be a model-complete theory. Then every $\mathfrak{n}$-orthoinjective is $\mathfrak{n}$-elementarily compact and so $\mathfrak{n}$-saturated.*

Proof. In T every formula is equivalent to an existential formula (this follows from Theorem 3.3.6 of [17]).

There are two applications worth mentioning.

First, consider the variety $\mathscr{B}$ of boolean algebras. Let B be the 8-element algebra with atoms a, b and c, and A the subalgebra of B generated by a and $b \vee c$. Let u: $\{0, 1\} \to A$ and v: $A \to B$ be the inclusions. Then it is easy to see that a non-trivial algebra C is $\{u, v\}$-orthoinjective iff C is *atomless*. Thus any non-trivial $\mathfrak{n}$-orthoinjective boolean algebra is an $\mathfrak{n}$-saturated model of the theory of atomless boolean algebras.

Second, let $\mathscr{L}$ be the full subcategory of $\mathscr{P}$ given by the totally ordered sets. Let $B = \{a, b, c, d, e\}$, where $a < b < c < d < e$, and $A = \{b, d\}$. Let u: $\{b\} \to A$ and v: $A \to B$ be the inclusions. Then a chain C is $\{u, v\}$-orthoinjective iff C is *dense* without endpoints. Thus an $\mathfrak{n}$-orthoinjective is an $\mathfrak{n}$-saturated model of the theory of dense linear order without end points.

We remark that an inductive theory with AP has enough $\mathfrak{n}$-orthoinjectives for all $\mathfrak{n}$, by using the method outlined in 6.10.

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