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UNDECIDABILITY OF THE EXISTENCE OF REGULAR EXTREMALLY DISCONNECTED S-SPACES

BY

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Following Rudin [5], a space is called an *S-space* if it is hereditarily separable but not Lindelöf. There are known examples of Hausdorff S-spaces, requiring nothing beyond ZFC (see [2] and [6]), but regular S-spaces are known to exist only under additional set-theoretic assumptions: Ostaszewski's example under Gödel's axiom of constructibility $V = L$ [4] or the recent example of Juhász et al. [3] under the continuum hypothesis (CH).

In this note we consider regular extremally disconnected S-spaces. The existence of such spaces is consistent with ZFC; Wage [7] has constructed an example of such a space under $V = L$, using Ostaszewski's method, and Ginsburg [1] has shown under Ostaszewski's (O) that such spaces exist in each infinite countably compact F-space. We give under (O′) (a condition slightly weaker than (O)), a method, distinct from that of Ginsburg [1], for a construction of such spaces in each infinite countably compact F-space.

On the other hand, we also show that if Martin's axiom and the negation of the continuum hypothesis (MA + $\neg$CH) hold, then a regular extremally disconnected S-space does not exist. In fact, we show that, under MA + $\neg$CH, each regular not hereditarily Lindelöf F-space contains an uncountable discrete subspace.

Throughout the paper, ordinals are denoted by small Greek letters, cardinals are initial ordinals, ω is the first infinite ordinal, ω_1 is the first uncountable ordinal, and $\beta\omega$ is the Čech-Stone compactification of an infinite countable discrete space. For undefined topological and cardinal notions we refer to [5].

1. An example using (O′). Ostaszewski [4] defines (O) as follows. Let $\{\lambda_\alpha: \alpha < \omega_1\}$ be the order-preserving indexing of limit ordinals in ω_1.

(O) *There exists a family* $\{S_\alpha\colon \alpha < \omega_1\}$ *of subsets of* ω_1 *such that* S_α *is a cofinal subset of* λ_α *and, if* S *is an uncountable subset of* ω_1, *then there exists an* $\alpha < \omega_1$ *with* $S_\alpha \subset S$.

If $V = L$ holds, then (O) holds; it is an easy consequence of Jensen's combinatorial statement $\diamondsuit$. In fact, K. Devlin has proved that (O)+CH is equivalent to $\diamondsuit$, while S. Shelah has shown that (O) does not imply $\diamondsuit$ (see [5]).

We define (O′) by replacing in (O) the condition "$S_\alpha \subset S$" by "$|S_\alpha - S| < \omega$". Trivially, (O′) is a consequence of (O), but we cannot prove the converse. **(P 1166)**

Now we show that, under (O′), $\beta\omega$ contains an extremally disconnected S-space.

Assume (O′) and let $\{\lambda_\alpha\colon \alpha < \omega_1\}$ and $\{S_\alpha\colon \alpha < \omega_1\}$ be as in (O′). Define $\{(D_\alpha, x_\alpha)\colon \alpha < \omega_1\}$ to fulfill the following conditions:

(1) D_α is a countable infinite discrete subspace of $\beta\omega - \omega$ and x_α is an arbitrarily chosen point from D_α;

(2) if $\alpha < \beta < \omega_1$, then $D_\beta \subset \mathrm{cl}\, D_\alpha - D_\alpha$;

(3) $D_{\lambda_\alpha} \subset \mathrm{cl}\{x_\xi\colon \xi \in S_\alpha\}$.

In the case of non-limit ordinal β the definition of (D_β, x_β) fulfilling (1) and (2) is obvious.

Now, suppose β is a limit ordinal, say $\beta = \lambda_\alpha$. Choose an arbitrary sequence $\xi_1 < \xi_2 < \ldots$ of ordinals from S_α cofinal with λ_α. Then the corresponding sequence $\{x_{\xi_n}\colon n < \omega\}$ is discrete and all its accumulation points are in $\bigcap\{\mathrm{cl}\, D_\xi - D_\xi\colon \xi < \lambda_\alpha\}$. Thus it is possible to choose a discrete infinite subset D_β in $\bigcap\{\mathrm{cl}\, D_\xi - D_\xi\colon \xi < \lambda_\alpha\} \cap \mathrm{cl}\{x_{\xi_n}\colon n < \omega\}$ which is therefore contained in $\mathrm{cl}\{x_\xi\colon \xi \in S_\alpha\}$.

Put $X = \omega \cup \{x_\alpha\colon \alpha < \omega_1\}$. Then X, as a dense subset of an extremally disconnected space $\beta\omega$, is regular and extremally disconnected.

X is hereditarily separable. Indeed, if Y is an uncountable subspace of X, then there exists an $\alpha < \omega_1$ such that

$$|\{x_\xi\colon \xi \in S_\alpha\} - Y \cap \{x_\alpha\colon \alpha < \omega_1\}| < \omega.$$

Hence, by (3), $Y \subset \mathrm{cl}\{x_\xi \in Y\colon \xi < \lambda_\alpha\} \cup \mathrm{cl}(Y \cap \omega)$. Let us note that if, in addition, Y is closed, then $\{x_\xi \in X\colon \xi \geqslant \lambda_\alpha\} \subset Y$. This shows that every closed subset of X is countable or its complement is countable.

X is not Lindelöf. Indeed, $\{(\omega^* - (\mathrm{cl}\, D_\alpha - D_\alpha)) \cap X\colon \alpha < \omega_1\}$ is an open cover of X without countable subcover.

Thus X is a regular extremally disconnected S-space. The properties stated for X imply that X is hereditarily normal, collectionwise normal, and perfectly normal. It may be also worthwhile to point out that the space X is scattered. Indeed, if $F \subset X - \omega$ is closed, then x_α, where $\alpha = \inf\{\xi\colon x_\xi \in F\}$, is an isolated point of F.

Remark. The construction of the space X under (O′) with the same properties and in the same manner as above is possible in any F-space which is countably compact.

2. MA $+\neg$CH and vanishing spaces. A space X is said to be *vanishing* if $X = \bigcup\{D_a: a < \omega_1\}$, where D_a is a countable infinite discrete subspace of X and $D_a \subset \mathrm{cl}D_\beta - D_\beta$ whenever $\beta < a$. Clearly, every vanishing space is not Lindelöf. So, every hereditarily separable vanishing space is an S-space.

THEOREM 1. *If Y is a T_1 not Lindelöf space in which discrete subspaces are at most countable, then Y contains a vanishing space.*

Proof. Let $\{U_\xi: \xi < \gamma\}$ be an open cover of Y without countable subcover. There exist $\{D_\xi: \xi < \omega_1\}$ and $\{S_\xi: \xi < \omega_1\}$ such that

(1) $\emptyset \neq D_\xi$ is discrete in Y, $S_\xi \subset \gamma$ and $|S_\xi| = \omega$ for each $\xi < \omega_1$;

(2) $D_\xi \subset \bigcup\{U_a: a \in S_\xi\}$ for each $\xi < \omega_1$;

(3) $\mathrm{cl}D_\xi \cup \bigcup\{U_a: a \in S_\xi\} \supset Y - \bigcup\big\{U_a: a \in \bigcup\{S_\nu: \nu < \xi\}\big\}$ for each $\xi < \omega_1$;

(4) $D_\xi \cap \bigcup\{U_a: a \in S_\beta\} = \emptyset$ for each $\beta < \xi < \omega_1$.

Before proving the existence of $\{D_\xi: \xi < \omega_1\}$ and $\{S_\xi: \xi < \omega_1\}$, observe that conditions (1)-(4) imply that $X = \bigcup\{D_\xi: \xi < \omega_1\}$ is a vanishing space. Indeed, from (1) and (2) we infer that each D_ξ is a countable discrete subspace of Y. So it remains to show that if $a < \beta < \omega_1$, then $D_\beta \subset \mathrm{cl}D_a - D_a$ (since each D_β is not empty, and each D_a is infinite). Let $a < \beta < \omega_1$. By (3),

$$\mathrm{cl}D_a \cup \bigcup\big\{U_\lambda: \lambda \in \bigcup\{S_\xi: \xi \leqslant a\}\big\} = Y.$$

By (2),

$$D_a \subset \bigcup\{U_\lambda: \lambda \in S_a\},$$

and, by (4),

$$D_\beta \cap \bigcup\big\{U_\lambda: \lambda \in \bigcup\{S_\xi: \xi \leqslant a\}\big\} = \emptyset.$$

Hence $D_\beta \subset \mathrm{cl}D_a - D_a$.

The zero-step in the inductive construction of families $\{D_\xi: \xi < \omega_1\}$ and $\{S_\xi: \xi < \omega_1\}$ fulfilling (1)-(4) is realized in the following way:

Choose an arbitrary point x_0 from U_0. Assume that points x_λ have been chosen for $\lambda < \xi$, $\xi < \gamma$. Then x_ξ is an arbitrary point from E_ξ,

$$E_\xi = U_\xi - \big(\bigcup\{U_\lambda: \lambda < \xi\} \cup \mathrm{cl}\{x_\lambda: \lambda < \xi\}\big),$$

if $E_\xi \neq \emptyset$, and x_ξ is an arbitrary point defined before if $E_\xi = \emptyset$.

The points $\{x_\xi: \xi < \gamma\}$ defined in such a way form a discrete subset of Y. To see this, let $\beta < \gamma$. Put $a = \inf\{\xi: x_\xi = x_\beta\}$. By the inductive assumption,

$$x_\beta = x_a \in U_a - \big(\bigcup\{U_\xi: \xi < a\} \cup \mathrm{cl}\{x_\xi: \xi < a\}\big).$$

Hence

$$\{x_\xi\colon\ \xi \leqslant a\} \cap (U_a - \mathrm{cl}\,\{x_\xi\colon\ \xi < a\}) = \{x_a\} = \{x_\beta\}.$$

However, each point x_ξ such that $x_\xi \neq x_\beta$ and $\xi > a$ lies outside the set U_a which contains $U_a - \mathrm{cl}\{x_\xi\colon\ \xi < a\}$. Hence

$$(U_a - \mathrm{cl}\{x_\xi\colon\ \xi < a\}) \cap \{x_\xi\colon\ \xi < \gamma\} = \{x_\beta\}.$$

Since discrete subsets of Y are at most countable, the set $\{x_\xi\colon\ \xi < \gamma\}$ is countable and we put this set to be D_0. Now we use D_0 for the definition of S_0.

Let $\{x^1, x^2, \ldots\}$ be an enumeration of D_0. For each $n < \omega$ let

$$a_n = \inf\{a < \gamma\colon\ x^n = x_a\}.$$

Put $S_0 = \{a_1, a_2, \ldots\}$. We verify that the required conditions are fulfilled by D_0 and S_0.

The verification of (1) and (2) is trivial, and (4) is empty. It remains to prove (3) which is here of the form

$$\mathrm{cl}\,D_0 \cup \bigcup\{U_\xi\colon\ \xi \in S_0\} = Y.$$

Since $\{U_\xi\colon\ \xi < \gamma\}$ is a cover of Y, it suffices to show that every member of it is contained in $\mathrm{cl}\,D_0 \cup \bigcup\{U_\xi\colon\ \xi \in S_0\}$.

Assume that it is not true. Then among members of that cover which are not contained in $\mathrm{cl}\,D_0 \cup \bigcup\{U_\xi\colon\ \xi \in S_0\}$ choose the one with the minimal index, say λ. Clearly, $\lambda \notin S_0$. But this means that

$$U_\lambda \subset \bigcup\{U_\xi\colon\ \xi < \lambda\} \cup \mathrm{cl}\,\{x_\xi\colon\ \xi < \lambda\} \subset \bigcup\{U_\xi\colon\ \xi < \lambda\} \cup \mathrm{cl}\,D_0.$$

However, by the minimality of λ,

$$U_\xi \subset \mathrm{cl}\,D_0 \cup \bigcup\{U_a\colon\ a \in S_0\} \quad \text{for each } \xi < \lambda,$$

and hence $U_\lambda \subset \mathrm{cl}\,D_0 \cup \bigcup\{U_\xi\colon\ \xi \in S_0\}$, a contradiction.

Thus the zero-step is complete.

Now, let us suppose that $\{D_\xi\colon\ \xi < a\}$ and $\{S_\xi\colon\ \xi < a\}$ are defined, where $a < \omega_1$. Since $\bigcup\{S_\xi\colon\ \xi < a\}$ is countable and $\{U_\xi\colon\ \xi < \gamma\}$ has no countable subcover, the set

$$Y' = Y - \bigcup\{U_\eta\colon\ \eta \in \bigcup\{S_\xi\colon\ \xi < a\}\}$$

is not empty and $\{U_\beta \cap Y'\colon\ \beta \in \gamma - \bigcup\{S_\xi\colon\ \xi < a\}\}$ is an open cover of Y'. Then, as in the zero-step, we can find a discrete subset $D_a \subset Y'$ and a countable set $S_a \subset \gamma - \bigcup\{S_\xi\colon\ \xi < a\}$ such that

$$Y' \subset \mathrm{cl}\,D_a \cup \bigcup\{U_\eta\colon\ \eta \in S_a\} \quad \text{and} \quad D_a \subset \bigcup\{U_\eta\colon\ \eta \in S_a\}.$$

It is easy to verify that $\{D_\xi\colon\ \xi \leqslant a\}$ and $\{S_\xi\colon\ \xi \leqslant a\}$ fulfill (1)-(4).

So we have the following reduction of the problem of the existence of S-spaces:

COROLLARY 1. *There exists a T_i S-space iff there exists a T_i hereditarily separable vanishing space* ($i = 1, 2, 3, 3\frac{1}{2}$).

The case of regular extremally disconnected vanishing spaces will be especially interesting for us. If MA + $\neg$CH holds, we have

THEOREM 2 (MA + $\neg$CH). *If X is a regular extremally disconnected vanishing space, then X contains an uncountable discrete subset.*

Proof. Suppose $X = \bigcup\{D_a\colon a < \omega_1\}$, where D_a is discrete in X and $D_\beta \subset \mathrm{cl}\,D_a - D_a$ for each $a < \beta < \omega_1$. Let $\{\lambda_a\colon a < \omega_1\}$ be the order-preserving indexing of the limit ordinals in ω_1. Now, for each $\xi < \omega_1$ we construct inductively a family R_ξ with the following properties:

(1) Each R_ξ consists of countably many subsets of $\bigcup\{D_a\colon a < \lambda_\xi\}$ and every two distinct members of R_ξ have finite intersection.

(2) If $\xi < \eta < \omega_1$, then $R_\xi \subset R_\eta$.

(3) If $F \subset \bigcup\{D_a\colon a < \lambda_\xi\}$ is such that F has finite intersections with each member of R_ξ and with each D_a, $a < \lambda_\xi$, then F is closed and discrete.

Let $\xi = 0$. Then R_0 is defined in the following way:

Since D_{ω_0} is countable and discrete, there exists a family $\mathscr{U}$, consisting of disjoint closed-open subsets of X, which separates the points of D_{ω_0}. Put

$$R_0 = \big\{U \cap \bigcup\{D_\xi\colon \xi < \omega_0\}\colon U \in \mathscr{U}\big\}.$$

Clearly, R_0 has properties (1) and (2). For the proof that R_0 has also property (3), let $F \subset \bigcup\{D_\xi\colon \xi < \omega_0\}$ be such that $|F \cap D_\xi| < \omega$ for each $\xi < \omega_0$ and $|F \cap U| < \omega$ for each $U \in \mathscr{U}$. From the first condition on F we get $\mathrm{cl}\,F - F \subset \bigcup\{D_\xi\colon \xi \geqslant \omega_0\}$, and thus, since $F \subset \bigcup\{D_\xi\colon \xi < \omega_0\}$, F is discrete. From the second condition on F we get $\mathrm{cl}\,F \cap D_{\omega_0} = \emptyset$. Since

$$\mathrm{cl}\,D_{\omega_0} = \bigcup\{D_\xi\colon \xi \geqslant \omega_0\} \quad \text{and} \quad F \subset \bigcup\{D_\xi\colon \xi < \omega_0\},$$

we have $F \cap \mathrm{cl}\,D_{\omega_0} = \emptyset$. That is, $\mathrm{cl}\,F \cap D_{\omega_0} = \emptyset = F \cap \mathrm{cl}\,D_{\omega_0}$. However, in such a situation, for countable sets in regular extremally disconnected spaces we have $\mathrm{cl}\,F \cap \mathrm{cl}\,D_{\omega_0} = \emptyset$. So, $\mathrm{cl}\,F - F = \emptyset$, which means that F is closed.

Assume that we have defined families D_ξ with properties (1)-(3) for each $\xi < a$, $a < \omega_1$. Let us consider now separately the case of non-limit and limit a.

In the case where a is a non-limit ordinal, say $a = \beta + 1$, we build, as in the zero-step, the family R_0' for $D_{\lambda_{\beta+1}}$ and $\{D_{\lambda_{\beta+n}}\colon n < \omega\}$ and we put $R_a = R_\beta \cup R_0'$. Clearly, the family R_a has properties (1) and (2). Now, if

$F \subset \bigcup\{D_\xi: \xi < \lambda_a\}$ has finite intersections with each member of R_a and with each D_ξ, $\xi < \lambda_a$, then $F = F_1 \cup F_2$, where

$$F_1 = F \cap \bigcup\{D_\xi:\ \xi < \lambda_\beta\} \quad \text{and} \quad F_2 = F \cap \bigcup\{D_\xi:\ \lambda_\beta \leqslant \xi < \lambda_{\beta+1}\}.$$

Both sets F_1 and F_2 are closed and discrete. Since they are also disjoint, F is closed and discrete. Thus R_a has also property (3).

Assume now that a is a limit ordinal. Let us put $R = \bigcup\{R_\eta:\ \eta < a\}$. Let a family $\mathcal{U}$, consisting of disjoint closed-open subsets of X, separate the points of the set D_{λ_a}. Let us choose (if it is possible) in each set $U \cap \bigcup\{D_\xi:\ \xi < \lambda_a\}$, where $U \in \mathcal{U}$, exactly one closed set F_U which has finite intersections with each member of the family R and the accumulation point in $U \cap D_{\lambda_a}$. Then we put

$$R_a = R \cup \{F_U:\ F_U \neq \emptyset,\ U \in \mathcal{U}\}.$$

Again, obviously, R_a has properties (1) and (2). For the proof of property (3), let us take a suitable F. Since $F \cap \bigcup\{D_\eta:\ \eta \leqslant \xi\}$ is closed and discrete for each $\xi < \lambda_a$, the structure on X assures that F is discrete and $\mathrm{cl}F - F$ is contained in $\mathrm{cl}D_{\lambda_a}$. We have $\mathrm{cl}F \cap D_{\lambda_a} = \emptyset$. For if $x \in \mathrm{cl}F \cap D_{\lambda_a}$, then we take $U \in \mathcal{U}$ with $U \cap D_{\lambda_a} = \{x\}$. Since $F \cap U$ is closed in $\bigcup\{D_\xi:\ \xi < \lambda_a\}$ and $F \cap U$ has finite intersections with each member of the family R and the accumulation point x, a non-empty set F_U also belongs to the family R_a. Since $|F \cap F_U| < \omega$, the sets $F' = F - F_U$ and F_U are countable, $\mathrm{cl}F' \cap F_U = \emptyset = F' \cap \mathrm{cl}F_U$, and $x \in \mathrm{cl}F' \cap \mathrm{cl}F_U$. But this is impossible in regular extremally disconnected spaces. For the proof of property (3), let us observe that since $\mathrm{cl}D_{\lambda_a} = \bigcup\{D_\xi:\ \xi \geqslant \lambda_a\}$, we have $F \cap \mathrm{cl}D_{\lambda_a} = \emptyset$. Since X is regular and extremally disconnected, $\mathrm{cl}F \cap \mathrm{cl}D_{\lambda_a} = \emptyset$ and, therefore, $\mathrm{cl}F - F = \emptyset$, i.e. F is closed.

Thus the construction of the families $\{R_a:\ a < \omega_1\}$ is complete.

Now, let us consider the family $R = \bigcup\{R_a:\ a < \omega_1\}$. The cardinality of R is ω_1 and, by (1) and (2), R consists of countable subsets of the set X, $|X| = \omega_1$. Moreover, every two different members of R have finite intersection. Under MA$+\neg$CH, there exists an uncountable set $M \subset X$ which has finite intersection with each member of R [7]. Without loss of generality we may assume that M has also finite intersection with each D_a, $a < \omega_1$.

The set M is discrete (and closed). This is a consequence of (3) and of a special structure of X. For the proof of this fact we assume, to the contrary, that there is $x \in M$ with $x \in \mathrm{cl}(M - \{x\})$. Let a be the minimal index among those indices ξ for which $x \in D_\xi$. Since $D_\xi \subset \mathrm{cl}D_a - D_a$ for each $\xi > a$ and D_a is discrete, x is an accumulation point of the set $F = M \cap \bigcup\{D_\xi:\ \xi < a+1\}$, that is, F is not discrete. However, F as a subset of M has finite intersection with each member of the family R_{a+1} and with each D_ξ. Thus, by (3), F is discrete, a contradiction.

It is well known that separable subspaces of regular F-spaces are regular and extremally disconnected. Thus, any vanishing subspace of a regular F-space is regular and extremally disconnected. Hence, from Theorems 2 and 1 we get the following corollary which was suggested to the author by Jan van Mill.

COROLLARY 2 ($\mathrm{MA}+\neg\mathrm{CH}$). *Any regular not hereditarily Lindelöf F-space contains an uncountable discrete subspace.*

In particular, we have

COROLLARY 3 ($\mathrm{MA}+\neg\mathrm{CH}$). *There is no regular extremally disconnected S-space.*

In [5] Rudin asks whether $\mathrm{MA}+\neg\mathrm{CH}$ implies that there is no regular S-space. Corollary 3 gives an answer for the class of extremally disconnected spaces.

In connection with Corollary 2 the following problem arises:

PROBLEM (**P 1167**). Does there exist an uncountable hereditarily separable and hereditarily Lindelöf regular extremally disconnected space?

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