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CESÀRO-PERRON-STIELTJES INTEGRAL

BY

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Introduction. Ward has defined the Perron-Stieltjes integral in [11]. We have introduced the definition of the Perron-Stieltjes integral by taking into consideration ω-derivative and ω-measure [7]. Burkill has defined the Cesàro-Perron integral in [2]. We have defined the Cesàro-continuity, the Cesàro-derivative of a function relative to ω. Following Burkill, we have defined the Cesàro-Perron-Stieltjes integral. Following Ward, Kubota [9] has defined the Cesàro-Perron-Stieltjes integral by the help of special Denjoy integral, though he has not established any relationship between his integral and Ward's integral. Our integral of this paper is the direct generalization of the Perron-Stieltjes integral and the Lebesgue-Stieltjes integral [7]. However, the integrals of Kubota and ourselves are different. Both integrals are the generalization of the Cesàro-Perron integral of Burkill (1). Further, we have established some important properties of the integral which Kubota has not done and, in particular, it is shown that if a function $f(x)$ is non-negative and integrable in our sense, then it is equal to the Lebesgue-Stieltjes (LS) integral.

1. Notation, definition and theorem. Let $\omega(x)$ be a non-decreasing function defined on the closed interval $[a, b]$. We extend the definition to all x by taking $\omega(x) = \omega(a)$ for $x < a$ and $\omega(x) = \omega(b)$ for $x > b$. Let S denote the set of points of continuity of $\omega(x)$ and $D = [a, b] - S$. Let S_0 denote the union of pairwise disjoint open intervals (a_i, b_i) in $[a, b]$ on each of which $\omega(x)$ is constant,

$$S_1 = \{a_1, b_1, a_2, b_2, \ldots\}, \quad S_2 = S \cap S_1 \quad \text{and} \quad S_3 = [a, b] \cap S - (S_0 \cup S_2).$$

Further, let S_2^- and S_2^+ denote those points of S_2 which are correspondingly left-end points and right-end points of (a_i, b_i). Jeffery [7] has denoted by $\mathcal{U}$ the class of functions $f(x)$ defined as follows: $f(x)$ is defined on the set $[a, b] \cap S$ such that $f(x)$ is continuous at every point of $[a, b] \cap S$ with respect to S. If $x_0 \in D$, then $f(x)$ tends to a limit as x tends to x_0+ or x_0- over the points of S. These limits are denoted by $f(x_0+)$ and $f(x_0-)$,

(1) Kubota has considered finite functions for the generalization.

respectively. Also, $f(x) = f(a+)$ for $x < a$ and $f(x) = f(b-)$ for $x > b$. The functions $f(x)$ may or may not be defined at the points of D. Suppose that $\mathscr{U}_0 \subset \mathscr{U}$ contains those functions $f(x)$ in $\mathscr{U}$ for which both $f(x_0+)$ and $f(x_0-)$ are finite for $x_0 \in D$. If a property P is satisfied at all points of a set A except a set of ω-measure zero (see [7] and [3]), then it is said that *P is satisfied ω-almost everywhere in A or at ω-almost all points of A.*

Definition 1.1 (see [7] and [3]). Let $f(x)$ belong to the class $\mathscr{U}_0$. For any x and $h \neq 0$ with $x+h \in S$ the function $\psi(x, h)$ is defined by

$$\psi(x, h) = \begin{cases} \dfrac{f(x+h)-f(x-)}{\omega(x+h)-\omega(x-)}, & h > 0,\ \omega(x+h)-\omega(x-) \neq 0, \\ \dfrac{f(x+h)-f(x+)}{\omega(x+h)-\omega(x+)}, & h < 0,\ \omega(x+h)-\omega(x+) \neq 0, \\ 0, & \omega(x+h)-\omega(x\pm) = 0. \end{cases}$$

If $\psi(x, h)$ tends to a limit as $h \to 0$ $(x+h \in S)$, then this limit is the *ω-derivative* of $f(x)$ at x and is denoted by $f'_\omega(x)$. *Right-hand ω-derivatives* $D^+f_\omega(x)$ and $D_+f_\omega(x)$ and *left-hand ω-derivatives $D^-f_\omega(x)$ and $D_-f_\omega(x)$* and, finally, *two-sided ω-derivatives $\overline{D}f_\omega(x)$ and $\underline{D}f_\omega(x)$* are defined in the usual way. Further, $f'_{-\omega}(x)$ and $f'_{+\omega}(x)$ will be used to denote the left-hand and right-hand ω-derivatives, respectively, of $f(x)$ at x.

THEOREM 1.1 (Theorem 2.3, [5]). *If $f(x) \in \mathscr{U}_0$ is* AC-ω *below* [1] *on* $[a, b]$, *then, on* $[a, b] \cap S$, *$f(x)$ can uniquely be represented in the form* $f(x) = \varphi(x) + r(x)$, *where $\varphi(x)$ is* AC-ω ([7], [1]) *on* $[a, b]$, $\varphi(a+) = f(a+)$, *$r(x)$ is continuous and non-decreasing on* $[a, b]$, *and* $r'_\omega(x) = 0$ *ω-almost everywhere in* $[a, b]$.

2. The PS-integral, (ω) C-continuity and (ω) C-derivative.

LEMMA 2.1. *Let $f(x) \in \mathscr{U}_0$ be non-decreasing on each $(a_i, b_i) \subset [a, b]$, where $\omega(x)$ is constant. If one of the four ω-derivatives of $f(x)$ is non-negative in $[a, b] \cap S$ and if $f'_\omega(x) \geqslant 0$ for $x \in D$, then $f(x)$ is non-decreasing on $[a, b] \cap S$.*

The proof is similar to that of Lemma 2.12 at the end of this section.

Let $f(x)$ be a function (which is not necessarily finite) defined on the closed interval $[a, b]$.

Definition 2.1. A function $M(x) \in \mathscr{U}_0$ is said to be a *major function* of $f(x)$ on $[a, b]$ if

(a) $M(x)$ is non-decreasing on each open interval $(a_i, b_i) \subset [a, b]$, where $\omega(x)$ is constant,

(b) $M(a-) = 0$,

(c) $D_-M_\omega(x) > -\infty$ on $S_3 \cup S_2^-$ and $D_+M_\omega(x) > -\infty$ on $S_3 \cup S_2^+$,

(d) $M'_\omega(x) \geqslant f(x)$ on D, $D_-M_\omega(x) \geqslant f(x)$ on $S_3 \cup S_2^-$, and $D_+M_\omega(x) \geqslant f(x)$ on $S_3 \cup S_2^+$.

A *minor function* $m(x)$ of $f(x)$ is defined in an analogous way.

By Lemma 2.1, we see that $M(x)-m(x)$ is non-decreasing on $[a, b]\cap S$. It follows that if $f(x)$ has major and minor functions, then $I(b) = \inf\{M(b+)\}$, and $J(b) = \sup\{m(b+)\}$ are finite and $I(b) \geqslant J(b)$.

Definition 2.2. A function $f(x)$ defined on the closed interval $[a, b]$ is said to be *integrable in the Perron-Stieltjes sense* (or to be PS-*integrable*) on this interval if

(1) it has at least one major function $M(x)$ and at least one minor function $m(x)$,

(2) $I(b) = J(b)$.

If $f(x)$ is PS-integrable on the closed interval $[a, b]$, then the common value $I(b) = J(b)$ is called the *Perron-Stieltjes integral* (or PS-*integral*) of the function on the closed interval $[a, b]$ and is denoted by

$$(\mathrm{PS})\int_a^b f(x)\,d\omega.$$

As an immediate consequence we get the following

LEMMA 2.2. *If a function $F(x)\in\mathscr{U}_0$ has a finite ω-derivative $f(x)$ everywhere in $[a,b]-S_0$ and if $F(x)$ is constant on each open interval $(a_i, b_i)\subsetneq S_0$, then $f(x)$ is* PS-*integrable on $[a, b]$ and*

$$F(b+)-F(a-) = (\mathrm{PS})\int_a^b f(x)\,d\omega.$$

LEMMA 2.3. *Given a set E of ω-measure zero in $S_3\cup S_2$ and any positive number ε, there exists a function $\mu(x)$ in the class $\mathscr{U}_0$ such that $\mu(x)$ is non-decreasing on $[a, b]$ and*

$$\mu(a-) = 0, \quad \mu(b+) < \varepsilon,$$
$$D_-\mu_\omega(x) = +\infty \quad \textit{for } x\in E\cap S_3\cup E\cap S_2^-,$$
$$D_+\mu_\omega(x) = +\infty \quad \textit{for } x\in E\cap S_3\cup E\cap S_2^+.$$

Proof. For every natural number n, let G_n be a bounded open set such that $G_n\supset E$ and $|G_n|_\omega<\varepsilon/4^n$. Let

$$\psi_n(x) = \begin{cases} 0 & \text{for } x<a,\\ |G_n\cap[a, x]|_\omega & \text{for } a\leqslant x\leqslant b,\\ \psi_n(b-) & \text{for } x>b.\end{cases}$$

Then the function

$$\mu(x) = \sum_{n=1}^{\infty}\psi_n(x)$$

is in the class $\mathscr{U}_0$ and is non-decreasing on $[a, b]$. Obviously, $\mu(a-) = 0$ and $\mu(b+)<\varepsilon$. If $x\in E-S_2^+$ and h ($h>0$) is sufficiently small, then the whole interval $[x, x+h]$ $(x+h \in S)$ lies in G_n for a certain fixed n. For such an h, we have

$$\psi_n(x+h) = \psi_n(x) + \{\omega(x+h) - \omega(x)\}.$$

Thus

$$\frac{\psi_n(x+h) - \psi_n(x)}{\omega(x+h) - \omega(x)} = 1.$$

Hence, if h is sufficiently small and N is any positive integer, then

$$\frac{\mu(x+h) - \mu(x)}{\omega(x+h) - \omega(x)} \geqslant N.$$

This gives $D_+\mu_\omega(x) = +\infty$. Similarly, we can show that if $x \in E - S_2^-$, then $D_-\mu_\omega(x) = +\infty$. This completes the proof of the lemma.

Using Lemma 2.3 it can be shown that the scope of the PS-integral remains unaltered if inequality (d) in the definition of the PS-major function and the corresponding inequality for the minor function are assumed to hold ω-almost everywhere on $[a, b] - S_0$.

One can verify that the fundamental properties which are given in Theorems 1, 4, 5 and 6 of Section 3, Chapter 16 in [10], are true corresponding to the Perron-Stieltjes integral and ω-measure. Further, if

$$(\mathrm{PS}) \int_a^b f(x)\,d\omega$$

exists and if $a < c < b$, then both

$$(\mathrm{PS}) \int_a^c f(x)\,d\omega \quad \text{and} \quad (\mathrm{PS}) \int_c^b f(x)\,d\omega$$

exist; if $c \in (a, b) \cap S$, then

$$(\mathrm{PS}) \int_a^b f(x)\,d\omega = (\mathrm{PS}) \int_a^c f(x)\,d\omega + (\mathrm{PS}) \int_c^b f(x)\,d\omega.$$

Definition 2.3. If the function $f(x)$ is defined and PS-integrable on $[a, b]$, then the function

$$F(x) = \begin{cases} 0 & \text{for } x < a, \\ (\mathrm{PS}) \int_a^x f(t)\,d\omega & \text{for } a \leqslant x \leqslant b, \\ F(b-) & \text{for } x > b \end{cases}$$

is called the *indefinite Perron-Stieltjes integral* of $f(x)$.

It can easily be proved that $F(x)$ is continuous on $[a, b] \cap S$ (cf. Theorem 1, Section 4, Chapter 16, [10]).

LEMMA 2.4. *If* $f(x) \in \mathscr{U}_0$ *is* AC-ω *below on* $[a, b]$, *then*

$$(\mathrm{LS}) \int_a^b f'_\omega(x)\,d\omega \leqslant f(b+)-f(a-).$$

Proof. By Theorem 1.1, we can write $f(x) = \varphi(x)+r(x)$, $x \in [a, b]\cap S_r$, where $\varphi(x)$ is AC-ω on $[a, b]$, $\varphi(a+) = f(a+)$, $r(x)$ is non-decreasing on $[a, b]$, and $r'_\omega(x) = 0$ ω-almost everywhere on $[a, b]$. Also, by Theorem 3 in [1], $f(x)$ is BV-ω on $[a, b]$, and so $f'_\omega(x)$ is summable (LS) [3] on $[a, b]$. Hence

$$\begin{aligned}(\mathrm{LS}) \int_a^b f'_\omega(x)\,d\omega &= (\mathrm{LS}) \int_a^b \varphi'_\omega(x)\,d\omega \\ &= \varphi(b+)-\varphi(a-) \qquad \text{(Theorems 4.1 and 4.2, [6])} \\ &\leqslant f(b+)-f(a-).\end{aligned}$$

This completes the proof.

LEMMA 2.5. *If* $F(x)$ *is the indefinite* PS-*integral of the function* $f(x)$ *on* $[a, b]$, *then* $F'_\omega(x) = f(x)$ ω-*almost everywhere in* $[a, b]$.

Proof. We first show that

$$\underline{D}F_\omega(x) \geqslant f(x) \tag{1}$$

ω-almost everywhere in $[a, b]$. If relation (1) is not true, then there are a positive number p and a set $E \subset S_3 \cup D$ of positive outer ω-measure μ such that

$$f(x)-\underline{D}F_\omega(x) > p \quad \text{for all } x \in E. \tag{2}$$

Choose ε arbitrarily such that $0 < \varepsilon < \frac{1}{2}\mu p$ and let $M(x)$ be a major function of $f(x)$ such that

$$M(b+)-F(b+) < \varepsilon. \tag{3}$$

Put $R(x) = M(x)-F(x)$. Then $R(x)$ is non-decreasing on $[a, b]\cap S_r$ and so AC-ω below on $[a, b]$. Hence, by Lemma 2.4 and by (3),

$$(\mathrm{LS}) \int_a^b R'_\omega(x)\,d\omega \leqslant R(b+) < \varepsilon.$$

So the set in which $R'_\omega(x) > \frac{1}{2}p$ has ω-measure less than μ. Thus there is a set E_1 of positive outer ω-measure contained in E in which $R'_\omega(x)$ exists and lies between 0 and $\frac{1}{2}p$. At a point x of E_1, $f(x) \leqslant \underline{D}F_\omega(x)+\frac{1}{2}p$, which contradicts relation (2). Hence $\underline{D}F_\omega(x) \geqslant f(x)$ ω-almost everywhere in $[a, b]$. Introducing a minor function it can be proved in a similar way that $\overline{D}F_\omega(x) \leqslant f(x)$ ω-almost everywhere in $[a, b]$. This proves the lemma.

LEMMA 2.6. *If a function $f(x)$ is summable* (LS) *on $[a, b]$, then it is also* PS-*integrable on $[a, b]$ and*

$$(\mathrm{PS})\int_a^b f(x)\,d\omega = (\mathrm{LS})\int_a^b f(x)\,d\omega.$$

The proof of this lemma is similar to that of Theorem 1 of Section 5, Chapter 16 in [10].

Definition 2.4. Let a real function $F(x)$ be defined finitely on $[a, b]$ and let it be PS-integrable on $[a, b]$. We say that $F(x)$ is *Cesàro-continuous relative* to ω or (ω) C-*continuous* at x_0 if

$$\lim_{\substack{h\to 0\\ x_0+h\in S}} (\omega)\mathrm{C}(F; x_0, x_0+h) = F(x_0),$$

where

$$(\omega)\mathrm{C}(F; x_0, x_0+h)$$
$$=\begin{cases}\dfrac{1}{\omega(x_0+h)-\omega(x_0-)}\displaystyle\int_{[x_0,x_0+h]} F(t)\,d\omega, & h>0,\ \omega(x_0+h)-\omega(x_0-)\neq 0,\\[2ex] \dfrac{1}{\omega(x_0+h)-\omega(x_0+)}\displaystyle\int_{[x_0,x_0+h]} F(t)\,d\omega, & h<0,\ \omega(x_0+h)-\omega(x_0+)\neq 0,\\[2ex] F(x_0+h), & \omega(x_0+h)-\omega(x_0\mp) = 0.\end{cases}$$

We note that $F(x)$ is (ω)C-continuous at each point $x \in D$. We denote by $\mathscr{U}_1$ the class of those functions $F(x)$ which have the following properties:

(i) $F(x)$ is defined finitely on $[a, b]$ so that $F(x)$ is PS-integrable on $[a, b]$;

(ii) at each point x_0 of D, $F(x)$ tends to a finite limit as $x\to x_0+$ or $x\to x_0-$ over the points of the set S and at x_0, $F(x)$ has the value $\frac{1}{2}[F(x_0+)+F(x_0-)]$;

(iii) $F(x) = F(a)$ for $x < a$ and $F(x) = F(b)$ for $x > b$.

Definition 2.5. Let $F(x)\in\mathscr{U}_1$. For a point x of the set S and for $h\neq 0$ with $x+h\in S$ the function $\varphi(x, h)$ is defined by

$$\varphi(x, h) = \begin{cases}\dfrac{(\omega)\mathrm{C}(F; x, x+h)-F(x)}{\frac{1}{2}\{\omega(x+h)-\omega(x)\}}, & \omega(x+h)-\omega(x)\neq 0,\\[2ex] 0, & \omega(x+h)-\omega(x) = 0.\end{cases}$$

If $\varphi(x, h)$ tends to a limit as $h\to 0$ $(x+h\in S)$, then this limit is called the (ω)C-*derivative* of $F(x)$ at x and is denoted by $\mathrm{C}DF_\omega(x)$. *Right-hand (ω)*C-*derivatives* $\mathrm{C}D^+F_\omega(x)$ *and* $\mathrm{C}D_+F_\omega(x)$ and *left-hand* (ω)C-*derivatives*

$CD^-F_\omega(x)$ and $CD_-F_\omega(x)$ and, finally, *two-sided* (ω)C-*derivatives* $\overline{CDF}_\omega(x)$ and $\underline{CDF}_\omega(x)$ are defined in the usual way. $CDF_{-\omega}(x)$ and $CDF_{+\omega}(x)$ will be used to denote the left-hand and right-hand (ω)C-derivatives, respectively, of $F(x)$ at x.

LEMMA 2.7. *If $F(x)$ is (ω)C-continuous and monotonic on the interval $[a, b]$, then $F(x)$ is continuous on $[a, b]\cap S$ with respect to S.*

Proof. We prove the lemma by considering the case where $F(x)$ is non-decreasing on $[a, b]$. Let $x \in S_3\cup S_2^+$; then

$$\text{(LS)} \int_{[x,x+h]} F(x+)d\omega \leqslant \text{(LS)} \int_{[x,x+h]} F(t)d\omega, \qquad x+h \in S \ (h>0).$$

So $F(x+) \leqslant (\omega)C(F; x, x+h)$. Taking limits as $h \to 0+$ $(x+h \in S)$, we get $F(x+) \leqslant F(x)$. Similarly, for a point x of $S_3\cup S_2^-$, we get $F(x-) \geqslant F(x)$. Hence $F(x)$ is continuous on $[a, b]\cap S$ with respect to S. This completes the proof of the lemma.

LEMMA 2.8. *If the sequence of functions $\{s_n(x)\}$ defined on $[a, b]$ converges uniformly to $s(x)$ in $[a, b]$ and is such that, for each n, $s_n(x)$ is (ω)C-continuous on $[a, b]$, then the limit function $s(x)$ is (ω)C-continuous on $[a, b]$.*

Proof. Choose $\varepsilon > 0$ arbitrarily. Fix an integer n_ε so that, for all x in $[a, b]$, $|s_n(x)-s(x)| < \varepsilon/3$ if $n > n_\varepsilon$. Let x_0 be a point of $[a, b]$. Then for any such n and for $x_0+h \in [a, b]\cap S$ we get

$$|(\omega)C(s_n; x_0, x_0+h)-(\omega)C(s; x_0, x_0+h)| \leqslant \varepsilon/3.$$

Fix an integer $n > n_\varepsilon$ and then fix $\delta > 0$ such that

$$|(\omega)C(s_n; x_0, x_0+h)-s_n(x_0)| < \varepsilon/3 \quad \text{if } |h| < \delta \text{ and } x_0+h \in [a, b]\cap S.$$

Then

$$|(\omega)C(s; x_0, x_0+h)-s(x_0)| < \varepsilon \quad \text{if } |h| < \delta \text{ and } x_0+h \in [a, b]\cap S.$$

Hence $s(x)$ is (ω)C-continuous on $[a, b]$. This proves the lemma.

Definition 2.6. Let a function $g(x)$ be defined on a set $A \subset S_3$ and let x_0 be a point of ω-density (Definition 3.1, [3]) of A. If

$$\lim_{x\to x_0} g(x) = g(x_0)$$

and $x \in A$ except for a set of points of ω-density zero at x_0, then $g(x)$ is said to be *approximately continuous* on A at x_0 relative to ω or *ω-approximately continuous* on A at x_0.

LEMMA 2.9. *If a set $A \subset S_3$ is ω-measurable and a function $g(x)$ is ω-measurable on A, then $g(x)$ is ω-approximately continuous at ω-almost all points of A.*

The proof of this lemma is analogous to that of Theorem 5.9 in [8].

If $F(x)$ is in the class $\mathscr{U}_1$, then $F(x)$ is PS-integrable on $[a, b]$, and so ω-measurable (Theorem 4.1, [3]) on $[a, b]$. Hence, proceeding in a way analogous to the method of the proof of Theorem 7.1 in [8] and using Lemma 2.9, we get

LEMMA 2.10. *If $F(x)$ is in the class $\mathscr{U}_1$, then the four (ω)C-derivatives of $F(x)$ are ω-measurable on $[a, b]\cap S$.*

LEMMA 2.11. *Let $F(x) \in \mathscr{U}_0$ be such that, at every point x_i of the set D,*

$$F(x_i) = \tfrac{1}{2}[F(x_i+)+F(x_i-)].$$

Then $F(x)$ belongs to the class $\mathscr{U}_1$ and is (ω)C-continuous on $[a, b]-D$. Further

$$CDF_{-\omega}(x) = F'_{-\omega}(x) \quad \text{for } x \in S_3\cup S_2^-,$$

$$CDF_{+\omega}(x) = F'_{+\omega}(x) \quad \text{for } x \in S_3\cup S_2^+$$

provided the right-hand members exist.

Proof. Since the function $F(x) \in \mathscr{U}_0$, $F(x)$ is ω-measurable on $[a, b]$. Further $F(x)$ is bounded on $[a, b]$, and so it is summable (LS) on $[a, b]$. So $F(x)$ belongs to the class $\mathscr{U}_1$. Let $x_0 \in S_3\cup S_2^+$. Then for $\varepsilon > 0$ chosen arbitrarily we can find a point x' $(x' > x_0)$ of S sufficiently close to x_0 such that, for all $x \in (x_0, x']\cap S$,

$$F(x_0)-\varepsilon < (\omega)C(F; x_0, x) < F(x_0)+\varepsilon.$$

Hence

$$\lim_{\substack{x\to x_0+\\ x\in S}} (\omega)C(F; x_0, x) = F(x_0).$$

In a similar way we can show that for a point $x_0 \in S_3\cup S_2^-$

$$\lim_{\substack{x\to x_0-\\ x\in S}} (\omega)C(F; x_0, x) = F(x_0).$$

Now, $F(x)$ is (ω)C-continuous on $[a, b]-D$. If $a \in S_3\cup S_2^+$ and $F'_{+\omega}(x)$ is finite, then for a point a_1 $(a_1 > a)$ of S sufficiently close to a and for all x in $(a, a_1]\cap S$ we have

$$-\varepsilon < \frac{F(x)-F(a)}{\omega(x)-\omega(a)} - F'_{+\omega}(a) < \varepsilon,$$

where $\varepsilon > 0$ is arbitrarily small. Then

$$-\varepsilon \leqslant \frac{(\omega)C(F; a, x)-F(a)}{\tfrac{1}{2}\{\omega(x)-\omega(a)\}} - F'_{+\omega}(a) \leqslant \varepsilon,$$

which shows that

$$CDF_{+\omega}(a) = F'_{+\omega}(a). \tag{4}$$

If $F'_{+\omega}(a)$ is infinite, then in a similar way we can show that $CDF_{+\omega}(a)$ is infinite and that (4) holds. Similarly we can show that if $\beta \in S_3 \cup S_2^-$, then $CDF_{-\omega}(\beta) = F'_{-\omega}(\beta)$. The proof of the lemma is now complete.

LEMMA 2.12. *Let $F(x) \in \mathcal{U}_1$ be such that*

(i) *$F(x)$ is (ω)C-continuous on $[a, b]-D$;*

(ii) *$F(x)$ is non-decreasing on each $(a_i, b_i) \subset S_0$;*

(iii) *$CD_+F_\omega(x) \geqslant 0$ for $x \in S_3 \cup S_2^+$;*

(iv) *$F(x_i+) \geqslant F(x_i-)$ for every $x_i \in D$.*

Then $F(x)$ is non-decreasing on $[a, b]$.

Proof. Let β $(\beta > a)$ be any point of the interval $[a, b]$. Choose $\varepsilon > 0$ arbitrarily. Then

$$(\omega)C(F; a, x) - F(a) \geqslant -\frac{\varepsilon}{2}[\omega(x) - \omega(a)] \tag{5}$$

(the strong inequality holds if $a \in D \cup S_3 \cup S_2^+$) for all $x \in (a, x_0] \cap S$, where x_0 is sufficiently near to a. Then in every neighbourhood on the right of a we can find a point x of S such that

$$F(x) - F(a) \geqslant -\varepsilon[\omega(x) - \omega(a)]. \tag{6}$$

If relation (6) does not hold for $a \in D \cup S_3 \cup S_2^+$, then there exists a point x_1 $(a < x_1 \leqslant x_0)$ of S such that for all x in $(a, x_1]$

$$F(x) - F(a) \leqslant -\varepsilon[\overline{\omega}(x) - \omega(a)],$$

where $\overline{\omega}(x) = \omega(x)$ if $x \in S$ and $\overline{\omega}(x) = \frac{1}{2}[\omega(x+) + \omega(x-)]$ if $x \in D$. In that case

$$(\omega)C(F; a, x_1) - F(a) = \frac{1}{\omega(x_1) - \omega(a)} \int_a^{x_1} [F(t) - F(a)]d\omega$$

$$\leqslant -\frac{\varepsilon}{\omega(x_1) - \omega(a)} \int_a^{x_1} [\overline{\omega}(t) - \omega(a)]d\omega = -\frac{\varepsilon}{2}[\omega(x_1) - \omega(a)],$$

which contradicts (5). Thus we can find a point x_1 of S sufficiently near to a $(x_1 > a)$ such that

$$F(x_1) - F(a) \geqslant -\varepsilon[\omega(x_1) - \omega(a)], \tag{7}$$

$$(\omega)C(F; a, x_1) - F(a) \geqslant -\frac{\varepsilon}{2}[\omega(x_1) - \omega(a)]. \tag{7'}$$

Similarly we can find a point x_2 $(x_2 > x_1)$ of S near to x_1 such that

$$F(x_2) - F(x_1) \geqslant -\varepsilon[\omega(x_2) - \omega(x_1)], \tag{8}$$

$$(\omega)C(F; x_1, x_2) - F(x_1) \geqslant -\frac{\varepsilon}{2}[\omega(x_2) - \omega(x_1)]. \tag{8'}$$

Proceeding in this way we obtain a strictly increasing sequence $\{x_n\}$ from S which tends to a limit ξ. Let x_m be any point of the sequence. Then

$$(9) \qquad F(x_m)-F(a) \geqslant -\varepsilon[\omega(x_m)-\omega(a)],$$

$$(10) \qquad (\omega)\mathrm{C}(F; x_m, x_n)-F(x_m) \geqslant -\frac{\varepsilon}{2}[\omega(x_n)-\omega(x_m)].$$

From (9) and (10) we get

$$(11) \quad (\omega)\mathrm{C}(F; x_m, x_n)-F(a) \geqslant -\varepsilon[\omega(x_m)-\omega(a)]-\frac{\varepsilon}{2}[\omega(x_n)-\omega(x_m)].$$

If $\xi \in D$, then by taking $m \to \infty$ in (9) we get

$$F(\xi-)-F(a) \geqslant -\varepsilon[\omega(\xi-)-\omega(a)],$$

and so

$$F(\xi)-F(a) \geqslant -\varepsilon[\omega(\xi)-\omega(a)].$$

Next suppose that $\xi \in S$; then from (11) we get, keeping x_m fixed,

$$(\omega)\mathrm{C}(F; x_m, \xi)-F(a) \geqslant -\varepsilon[\omega(x_m)-\omega(a)]-\frac{\varepsilon}{2}[\omega(\xi)-\omega(x_m)].$$

Since $F(x)$ is (ω)C-continuous at ξ, making $m \to \infty$ we get

$$F(\xi)-F(a) \geqslant -\varepsilon[\omega(\xi)-\omega(a)].$$

If $\xi < \beta$, then we cover the interval $[a, \beta]$ by a Lebesgue chain and thus obtain

$$F(\beta)-F(a) \geqslant -\varepsilon[\omega(\beta)-\omega(a)].$$

Since $\varepsilon > 0$ is arbitrary, this relation gives $F(\beta) \geqslant F(a)$, which proves the lemma.

3. Major and minor functions and the CPS-integral.

Definition 3.1. Let a function $f(x)$ be defined (not necessarily finite) on $[a, b]$. Then $M(x) \in \mathscr{U}_1$ is said to be a CPS-*major function* of $f(x)$ on $[a, b]$ if

(a) $M(x)$ is (ω)C-continuous on $[a, b]-D$,

(b) $M(a) = 0$,

(c) $M(x)$ is non-decreasing on each $(a_i, b_i) \subset S_0$,

(d) $\mathrm{C}D_-M_\omega(x) > -\infty$ for $x \in S_3 \cup S_2^-$ and $\mathrm{C}D_+M_\omega(x) > -\infty$ for $x \in S_3 \cup S_2^+$,

(e) $\mathrm{C}D_-M_\omega(x) \geqslant f(x)$ for $x \in S_3 \cup S_2^-$ and $\mathrm{C}D_+M_\omega(x) \geqslant f(x)$ for $x \in S_3 \cup S_2^+$,

(f) $M(x_i+)-M(x_i-) \geqslant f(x_i)[\omega(x+)-\omega(x_i-)]$ for every $x_i \in D$.

A CPS-*minor function* $m(x)$ is defined in an analogous way.

THEOREM 3.1. *If $M(x)$ is a* CPS-*major function and $m(x)$ is a* CPS-*minor function of $f(x)$ on $[a, b]$, then the difference $R(x) = M(x)-m(x)$ is non-decreasing on $[a, b]$.*

The theorem follows from Lemma 2.12.

If $f(x)$ has CPS-major and CPS-minor functions, then $I_0(b) = \inf\{M(b)\}$ and $J_0(b) = \sup\{m(b)\}$ are finite and $I_0(b) \geqslant J_0(b)$.

Definition 3.2. A function $f(x)$ defined on the closed interval $[a, b]$ is said to be *integrable in the Cesàro-Perron-Stieltjes sense* (or to be CPS-*integrable*) on this interval if

(1) it has at least one CPS-major function $M(x)$ and at least one CPS-minor function $m(x)$;

(2) $I_0(b) = J_0(b)$.

If $f(x)$ is CPS-integrable on the closed interval $[a, b]$, then the common value $I_0(b) = J_0(b)$ is called the *Cesàro-Perron-Stieltjes integral* (or CPS-*integral*) of the function on the closed interval $[a, b]$ and is denoted by

$$(\text{CPS})\int_a^b f(x)\,d\omega.$$

4. Properties of the CPS-integral. A direct consequence of the definition of the CPS-integral is the following

THEOREM 4.1. *Let a function $F(x) \in \mathscr{U}_1$ have a finite* (ω)C-*derivative $f(x)$ at all points of $[a, b]\cap S - S_0$. If $F(x)$ is constant on the intervals of $S_0 \cup S_2$ and if*

$$F(x_i+) - F(x_i-) = f(x_i)[\omega(x_i+) - \omega(x_i-)] \quad \textit{for every } x_i \in D,$$

then $f(x)$ is CPS-*integrable on $[a, b]$ and*

$$(\text{CPS})\int_a^b f(x)\,d\omega = F(b) - F(a).$$

In place of the function $\mu(x)$ considered in Lemma 2.3 we now consider the function $\bar{\mu}(x)$ such that

$$\bar{\mu}(x) = \begin{cases} \frac{1}{2}[\mu(x+) - \mu(x-)] & \text{if } x \in D, \\ \mu(x) & \text{if } x \in S. \end{cases}$$

By the help of Lemmas 2.3 and 2.11 it can now be proved that the scope of the CPS-integral is unaltered if inequalities (e) in Definition 3.1 of the CPS-major function and the corresponding inequalities for the CPS-minor function are assumed to hold ω-almost everywhere in $[a, b]\cap S - S_0$. As an immediate consequence we get the following

THEOREM 4.2. *If a function $f(x)$ is* CPS-*integrable on the closed interval $[a, b]$ and if $g(x) = f(x)$ ω-almost everywhere in $[a, b]$, then $g(x)$ is also* CPS-*integrable on $[a, b]$ and*

$$(\text{CPS})\int_a^b g(x)\,d\omega = (\text{CPS})\int_a^b f(x)\,d\omega.$$

We observe that the CPS-integral is generalization of the PS-integral, and so of the LS-integral [7]. Obviously, it is also generalization of the CP-integral [2], the P-integral and the L-integral. The next four theorems follow very simply from the definition of the CPS-integral.

THEOREM 4.3. *If each of the functions* $f_1(x)$ *and* $f_2(x)$ *is* CPS-*integrable on the closed interval* $[a, b]$, *then their sum is also* CPS-*integrable on this interval and*

$$(\mathrm{CPS})\int_a^b [f_1(x)+f_2(x)]d\omega = (\mathrm{CPS})\int_a^b f_1(x)d\omega + (\mathrm{CPS})\int_a^b f_2(x)d\omega.$$

THEOREM 4.4. *If* $f(x)$ *is* CPS-*integrable on* $[a, b]$ *and* K *is a finite constant, then the function* $Kf(x)$ *is also* CPS-*integrable on* $[a, b]$ *and*

$$(\mathrm{CPS})\int_a^b Kf(x)d\omega = K(\mathrm{CPS})\int_a^b f(x)d\omega.$$

THEOREM 4.5. *If* $f(x)$ *is* CPS-*integrable on the closed interval* $[a, b]$ *and if* $a < c < b$, *then* $f(x)$ *is* CPS-*integrable on each of the intervals* $[a, c]$ *and* $[c, b]$ *and*

$$(\mathrm{CPS})\int_a^b f(x)d\omega = (\mathrm{CPS})\int_a^c f(x)d\omega + (\mathrm{CPS})\int_c^b f(x)d\omega.$$

THEOREM 4.6. *A* CPS-*integrable function* $f(x)$ *is finite* ω-*almost everywhere in* $[a, b]$.

Definition 4.1. If the function $f(x)$ is defined and CPS-integrable on $[a, b]$, then the function

$$F(x) = \begin{cases} 0 & \text{for } x < a, \\ (\mathrm{CPS})\int_a^x f(x)d\omega & \text{for } a \leqslant x \leqslant b, \\ F(b) & \text{for } x > b \end{cases}$$

is called the *indefinite Cesàro-Perron-Stieltjes integral* of $f(x)$.

THEOREM 4.7. *The indefinite Cesàro-Perron-Stieltjes integral of* $f(x)$ *is* (ω)C-*continuous on* $[a, b]$.

The theorem can be proved by the help of Lemma 2.8.

THEOREM 4.8. *If* $f(x)$ *is* CPS-*integrable on* $[a, b]$, $F(x)$ *is its indefinite* CPS-*integral and* $M(x)$ *and* $m(x)$ *are a* CPS-*major function and a* CPS-*minor function of* $f(x)$, *respectively, then each of the differences* $M(x)-F(x)$ *and* $F(x)-m(x)$ *is non-decreasing on* $[a, b]$.

The proof is simple, and so is omitted.

THEOREM 4.9. *If* $F(x)$ *is the indefinite* CPS-*integral of the function* $f(x)$ *defined on* $[a, b]$, *then*

$$CDF_\omega(x) = f(x)$$

ω-almost everywhere in $[a, b]\cap S$. Further, for each $x_i \in D$

$$F(x_i+)-F(x_i-) = f(x_i)[\omega(x_i+)-\omega(x_i-)].$$

The proof of the first part of this theorem is exactly alike to that of Lemma 2.5. The second part follows directly from the definition of CPS-major and CPS-minor functions.

COROLLARY 4.1. *If $f(x)$ is* CPS-*integrable on $[a, b]$, then $f(x)$ is ω-measurable on $[a, b]$.*

The corollary follows from Theorem 4.9 and Lemma 2.10.

THEOREM 4.10. *If $f(x)$ is non-negative and* CPS-*integrable on $[a, b]$, then $f(x)$ is summable* (LS) *on $[a, b]$ and*

$$(\mathrm{LS})\int_a^b f(x)\,d\omega = (\mathrm{CPS})\int_a^b f(x)\,d\omega.$$

Proof. Let $M(x)$ be a CPS-major function of $f(x)$ on $[a, b]$. By Lemmas 2.7 and 2.12, $M(x)$ is in the class $\mathscr{U}_0$ and AC-ω below on $[a, b]$. Since $M'_\omega(x) \geqslant f(x)$ ω-almost everywhere in $[a, b]-S_0$, $f(x)$ is summable (LS) on $[a, b]$ and

$$\begin{aligned}(\mathrm{LS})\int_a^b f(x)\,d\omega \leqslant (\mathrm{LS})\int_a^b M'_\omega(x)\,d\omega \leqslant M(b+) \quad &\text{(by Lemma 2.4)}\\ = M(b).&\end{aligned}$$

Since this is true for any CPS-major function $M(x)$, we have

$$(\mathrm{LS})\int_a^b f(x)\,d\omega \leqslant (\mathrm{CPS})\int_a^b f(x)\,d\omega. \tag{12}$$

Again, if $M(x)$ is an LS-major function (Definition 2.1, [6]) for the given function such that $M(x) = \frac{1}{2}[M(x+)+M(x-)]$ at points of D, then $M(x)$ is also a CPS-major function, and so

$$(\mathrm{LS})\int_a^b f(x)\,d\omega \geqslant (\mathrm{CPS})\int_a^b f(x)\,d\omega. \tag{13}$$

Combining (12) and (13) we get

$$(\mathrm{LS})\int_a^b f(x)\,d\omega = (\mathrm{CPS})\int_a^b f(x)\,d\omega.$$

This completes the proof of the theorem.

THEOREM 4.11. *Let a sequence of functions $f_1(x), f_2(x), \ldots, f_n(x), \ldots$, converging pointwise to a function $f(x)$ be defined on $[a, b]$. If, for all n and x in $[a, b]$, $g(x) \leqslant f_n(x) \leqslant h(x)$, where $g(x)$, $f_n(x)$ and $h(x)$ are all* CPS-*integrable on $[a, b]$, then $f(x)$ is* CPS-*integrable on $[a, b]$ and*

$$\lim_{n\to\infty}(\mathrm{CPS})\int_a^b f_n(x)\,d\omega = (\mathrm{CPS})\int_a^b f(x)\,d\omega.$$

Proof. Put $\varphi_n(x) = f_n(x) - g(x)$, $\varphi(x) = f(x) - g(x)$, and $\psi(x) = h(x) - g(x)$. Then $\varphi_n(x)$ and $\psi(x)$ are non-negative on $[a, b]$ and so, by Theorem 4.10, they are LS-integrable on $[a, b]$. Since $0 \leqslant \varphi_n(x) \leqslant \psi(x)$ and $a \leqslant x \leqslant b$, we get

$$\lim_{n\to\infty} (\mathrm{LS}) \int_a^b \varphi_n(x)\,d\omega = (\mathrm{LS}) \int_a^b \varphi(x)\,d\omega.$$

Hence we obtain

$$\lim_{n\to\infty} (\mathrm{CPS}) \int_a^b f_n(x)\,d\omega = (\mathrm{CPS}) \int_a^b f(x)\,d\omega.$$

This completes the proof of the theorem.

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