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A PROOF OF J. MAŘÍK'S LEMMA

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In Mařík's paper [1], the proof of Lemma 1 is incorrect as Hewitt pointed out in the Mathematical Reviews [MR 19, p. 535-536]. However, it was not mentioned whether this lemma is true or not. Since the proof of Theorem 2 depends on this lemma and Theorem 3 is a consequence of Theorem 2 in his paper, it is worth showing where the proof of this lemma is not correct and proving its validity.

We begin with the statement of his Lemma 1:

If $\mathfrak{L}$ is a σ-algebra of subsets of A, ν is a finite measure defined on $\mathfrak{L}$, and $A = \bigcup_{n=1}^{\infty} A_n$, then there is a sequence of positive numbers $a_1, a_2, \ldots$ with $a_n \to \infty$ such that $\left|\int_A f d\nu\right| < 1$ whenever f is an $\mathfrak{L}$-measurable function satisfying conditions: $x \in A \Rightarrow |f(x)| < a_n$, $n = 1, 2, 3, \ldots$

The assertion in his proof that there is a sequence of numbers $0 = a_0 < a_1 < a_2 < \ldots$ with $a_n \to \infty$ and

$$\sum_{n=1}^{\infty} a_n\big(\nu(A) - \alpha_{n-1}\big) < 1 \qquad \Big(\alpha_n = \inf\big\{\nu(B)\colon\ B \in \mathfrak{L},\, B \supset \bigcup_{j=1}^{n} A_j\big\}\Big)$$

is not true as the following example shows. Let $A = (0, 1]$, ν = Lebesgue measure and $A_n = [1/n, 1]$. Thus we have

$$A = \bigcup_{n=1}^{\infty} A_n,$$

$$\alpha_n = \nu(A_n) = 1 - \frac{1}{n}.$$

Obviously, $\alpha_n \to \nu(A) = 1$, $\nu(A) - \alpha_0 = 1$, and $\nu(A) - \alpha_{n-1} = 1/(n-1)$, $n = 2, 3, \ldots$ But for any sequence of numbers $0 = a_0 < a_1 < a_2 < \ldots$ with $a_n \to \infty$, we obtain

$$\sum_{n=1}^{\infty} a_n\big(\nu(A) - a_{n-1}\big) = a_1 + \sum_{k=2}^{\infty} a_k\big(\nu(A) - a_{k-1}\big)$$

$$\geqslant a_1 + a_1 \sum_{k=2}^{\infty} \big(\nu(A) - a_{k-1}\big)$$

$$= a_1 + a_1 \sum_{k=2}^{\infty} \frac{1}{k-1} = \infty .$$

We now divide the proof into four steps.

Step 1. *If* $a_n = \inf\{\nu(B)\colon B \in \mathfrak{L}, B \supset \bigcup_{j=1}^{n} A_j\}$, *then* $a_n \to 1$ *for some real number* 1.

This follows easily from the hypothesis.

Step 2. *There is a sequence of sets* $\{H_n\} \subset \mathfrak{L}$ *such that* $H_n \subset H_{n+1}$, $H_n \supset \bigcup_{j=1}^{n} A_j$, *and* $\nu(H_n) = a_n$, $n = 1, 2, 3, \ldots$

Proof. By definition of a_n, for each n there is a set $H_n \in \mathfrak{L}$ such that $H_n \supset \bigcup_{j=1}^{n} A_j$ and $\nu(H_n) = a_n$. Moreover, we may assume $H_{n+1} \supset H_n$, $n = 1, 2, 3, \ldots$ For if $H_n \not\subset H_{n+1}$, we can use $H_{n+1} \cup H_n$ instead of H_{n+1} since

$$a_n = \nu(H_n) \geqslant \nu(H_n \cap H_{n+1}) \geqslant a_n$$

implies

$$\nu(H_{n+1} \cup H_n) = \nu(H_{n+1}) + \nu(H_n - H_{n+1}) = \nu(H_{n+1}) + \nu(H_n) - \nu(H_n \cap H_{n+1})$$
$$= \nu(H_{n+1}) = a_{n+1}.$$

Step 3. *There is a sequence of numbers* $0 = b_0 < b_1 < b_2 < \ldots$ *with* $b_n \to \infty$ *such that*

$$\int_A |f|\,d\nu < q \qquad \Big(q = \frac{1}{2}a_{n_1} + \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ for some positive integer } n_1\Big)$$

if f *is an* $\mathfrak{L}$*-measurable function with conditions*: $x \in A_n \Rightarrow |f(x)| < b_n$, $n = 1, 2, 3, \ldots$

Proof. By Step 1, for each positive integer k there exists n_k such that $m > n \geqslant n_k$ implies $0 \leqslant a_m - a_n < 1/k^3$. Furthermore, we may choose $\{n_k\}$ such that $1 + n_k < n_{k+1}$, $k = 1, 2, 3, \ldots$

Define $n_0 = 0$, $b_{n_0} = b_0 = 0$, $b_{n_1} = \frac{1}{2}$, and $b_{n_k} = k-1$, $k = 2, 3, \ldots$ Further for $n_k < i < n_{k+1}$, define b_i such that $b_{n_k} \leqslant b_n < b_m \leqslant b_{n_{k+1}}$ if $n_k \leqslant n < m \leqslant n_{k+1}$. (Equality holds only if $n_k = n$, $m = n_{k+1}$ respectively.) Clearly, $0 = b_0 < b_1 < b_2 < \ldots$ and $b_n \to \infty$. Let f be a function satisfying the conditions stated in Step 3. We show $\int_A |f|\,d\nu < q$.

Define $B_n = \{x \colon x \in H_n,\ |f(x)| < b_n\}$, $n = 1, 2, 3, \ldots$ Trivially, $B_{n+1} \supset B_n$ and $\lim_{n\to\infty} B_n = \bigcup_{n=1}^{\infty} B_n = A$. Also $\bigcup_{i=1}^{n} A_i \subset B_n \subset H_n$ implies $\nu(B_n) = a_n$. From this we have $\nu(B_n - B_{n-1}) = \nu(B_n) - \nu(B_{n-1}) = a_n - a_{n-1}$. By a well-known theorem ([2], p. 28),

$$\int_A |f|\,d\nu = \int_{\bigcup_{n=1}^{\infty} B_n} |f|\,d\nu = \int_{\bigcup_{n=1}^{\infty} (B_n - B_{n-1})} |f|\,d\nu$$

$$= \sum_{n=1}^{\infty} \int_{B_n - B_{n-1}} |f|\,d\nu \leqslant \sum_{n=1}^{\infty} b_n(a_n - a_{n-1})$$

$$\leqslant \sum_{n=1}^{n_1} b_n(a_n - a_{n-1}) + \sum_{k=1}^{\infty} \sum_{n=n_k+1}^{n_{k+1}} b_n(a_n - a_{n-1})$$

$$< b_{n_1} \sum_{n=1}^{n_1} (a_n - a_{n-1}) + \sum_{k=1}^{\infty} b_{n_{k+1}} \Big[\sum_{n=n_k+1}^{n_{k+1}} (a_n - a_{n-1}) \Big]$$

$$= \tfrac{1}{2} a_{n_1} + \sum_{k=1}^{\infty} k(a_{n_{k+1}} - a_{n_k}) < \tfrac{1}{2} a_{n_1} + \sum_{k=1}^{\infty} k \cdot \frac{1}{k^3} = \tfrac{1}{2} a_{n_1} + \sum_{k=1}^{\infty} \frac{1}{k^2} = q.$$

Step 4. *There is a sequence of numbers* $0 = a_0 < a_1 < a_2 < \ldots$ *with* $a_n \to \infty$ *such that* $\int_A |f|\,d\nu < 1$ *whenever* f *is an* $\mathfrak{L}$*-measurable function satisfying conditions:* $x \in A_n \Rightarrow |f(x)| < a_n$, $n = 1, 2, 3, \ldots$

Proof. Define $a_n = b_n/(q+1)$, where b_n, $n = 0, 1, 2, \ldots$, and q are found in Step 3. If f is an $\mathfrak{L}$-measurable function with conditions: $x \in A_n \Rightarrow |f(x)| < a_n$, $n = 1, 2, 3, \ldots$, then by repeating the above proof when b_n is replaced by a_n, we obtain

$$\int_{|A|} |f|\,d\nu < \frac{q}{q+1} < 1.$$

The lemma is proved.

REFERENCES

[1] J. Mařík, *Les fonctionnelles sur l'ensemble des fonctions continues bornées, définies dans un espace topologique*, Studia Mathematica 16 (1957), p. 86-94.

[2] S. Saks, *Theory of the integral*, Warszawa - Lwów 1937.

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