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A CHARACTERIZATION OF DIFFERENTIABLE SUBMANIFOLDS OF EUCLIDEAN SPACES

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As is well known since Whitney (cf. [3] and [4]) every differentiable manifold M^k of class C^r is diffeomorphic to a submanifold of the same class C^r of a Euclidean space R^n and, therefore, can be considered as a submanifold of R^n. The question arises under what conditions a manifold M^k of dimension k and of class C^r embedded (topologically) in R^n is a submanifold of the class C^r of R^n. In 1965, Gluck [2] has provided such conditions for $k = 1$ or 2 and $r = 1$, and it is the aim of the present note to show the general case.

We start with the list of notations:

GL_n — the general linear group of all linear automorphisms of R^n;

O_n — the group of all orthogonal transformations of R^n;

$G_{n,k}$ — Grassman manifold of k-planes in R^n;

$V^*_{n,k}$ — Stiefel manifold of k-frames (i. e., of systems of k linearly independent vectors) of R^n.

Since GL_n acts on $V^*_{n,k}$, $V^*_{n,k}$ can be defined as the homogeneous space $GL_n/L_{n,k}$, where $L_{n,k}$ denotes the stability subgroup of GL_n.

Before proceed to the characterization theorem we need four lemmas. The first two seem to be known but we could not find them in the literature.

LEMMA 1. *The manifolds $G_{n,k}$ and $GL_n/(L_{n,k}\times GL_k)$ are diffeomorphic to each other.*

Proof. Since the orthogonal group O_n acts on $GL_n/(L_{n,k}\times GL_k)$ by the formula $g[A] = [gA]$ $(g\in O_n, A\in GL_n)$, it suffices to show that the mapping μ^* tangent to

$$\mu\colon O_n\to GL_n/(L_{n,k}\times GL_k)\colon g\mapsto g[E] = [g]$$

is onto for each point of O_n, because then O_n acts with the maximal order, and so $O_n/(O_{n-k}\times O_k) = G_{n,k}$ is diffeomorphic to $GL_n/(L_{n,k}\times GL_k)$.

It is clear that $\mu = p\,|\,_{O_n}$, where p denotes the canonical mapping $p\colon GL_n\to GL_n/(L_{n,k}\times GL_k)$.

By the local triviality of p we have $p_*^{-1}\{0\} = T(L_{n,k}\times GL_k)$. Since $(L_{n,k}\times GL_k)\cap O_n = O_{n-k}\times O_k$, there is $\mu_*^{-1}\{0\} = T(O_{n-k}\times O_k)$, so that

$$\dim \mu_*^{-1}\{0\} = \dim T(O_{n-k}\times O_k) = \tfrac{1}{2}(n^2+2k^2-2nk-n)$$

and

$$\dim TO_n - \dim \mu_*^{-1}\{0\} = k(n-k) = \dim GL_n/(L_{n,k}\times GL_k).$$

Hence μ_* is onto.

LEMMA 2. *Stiefel manifold $V^*_{n,k}$ is a fibre space with the base $G_{n,k}$, fibre GL_k and the projection $\pi\colon V^*_{n,k}\to G_{n,k}$ of class C^∞, where $\pi(v)$ is the subspace spanned by the vectors of frame v.*

In fact, there is $V^*_{n,k} = GL_n/L_{n,k}$ and, by Lemma 1, we have $G_{n,k} = GL_n/(L_{n,k}\times GL_k)$. Hence $V^*_{n,k}$ must be the fibre space described in the lemma (cf. [1], p. 173).

LEMMA 3. *Let $\pi\colon V^*_{n,k}\to G_{n,k}$ be the projection defined in Lemma 2. If $p\in G_{n,k}$, $w^0 = \langle w_1^0, \ldots, w_k^0\rangle \in V^*_{n,k}$, and $\pi(w^0) = p$, then there exists a neighbourhood W of p and a cross-section*

$$w : W \to V^*_{n,k} : q \mapsto \langle w_1(q), \ldots, w_k(q)\rangle$$

of class C^∞ such that $w_i(q) = w_i^0 + F_i(q)$ and $F_i(q)\in p^\perp$, where $p^\perp$ denotes the orthogonal complement to p in R^n.

Proof. Since $\pi\colon V^*_{n,k}\to G_{n,k}$ is locally trivial, there exists a neighbourhood U of p and a cross-section $u^0\colon U^0\to V^*_{n,k}$ of class C^∞ such that $u^0(p) = w^0$. We improve u^0 by using the following simple property of frames:

If $f\colon M\to V^*_{n,k}\colon v\mapsto \langle f_1(x), \ldots, f_k(x)\rangle$ and $a\colon M\to R$ are both of the class C^∞, then $\hat{f} = \langle f_1, \ldots, f_{i-1}, f_i+f_j, f_{i+1}, \ldots, f_k\rangle$ $(i\neq j)$ is a function from M into $V^*_{n,k}$ of class C^∞ and $\pi f = \pi\hat{f}$.

Let $w_{k+1}^0, \ldots, w_n^0$ be a base of p. Writing

$$u_i^0 = \sum_{j=1}^{n} \lambda_{ij}^0(q)\, w_j^0,$$

we can represent cross-section u^0 by the matrix $\Lambda_0(q) = \big(\lambda_{ij}^0(q)\big)$. What we need, however, is a cross-section with the matrix $\Lambda = (\lambda_{ij})$ such that $\lambda_{ij}(q) = \delta_{ij}$ for $j\leqslant k$ and $q\in W$ (δ_{ij} here and below denotes Kronecker's symbol).

Since $\lambda_{11}^0(p) = 1$, we can multiply the first row of Λ_0 by $\lambda_{i1}^0/\lambda_{11}^0$ and subtract it from i-th row, $i = 2, 3, \ldots, k$. Dividing, in addition, the first row by λ_{11}^0, we obtain a new matrix $\Lambda_1 = (\lambda_{ij}^1)$ with the improved first column. The cross-section u^1 represented by Λ_1 is defined on a neighbourhood U^1 of p such that $U^1\subset U^0$ and $\lambda_{11}^0(q)\neq 0$ for $q\in U^1$. Since again $\lambda_{ij}^1(p) = \delta_{ij}$ for $j\leqslant k$, we can improve second column of Λ_1 using the second row, etc. It is easy to see that Λ_k has the needed form, hence it represent a cross-section u^k: $U^k \to V^*_{n,k}$ having the properties required in the lemma.

LEMMA 4. *If the graph of a function f: $R^k \to R^n$ has, for each point y, tangent hyperplane $P(y)$ such that the orthogonal projection π: $R^{n+k} \to R^k$ carries $P(y)$ onto R^k, then f is differentiable.*

Proof. Fix $x_0 \in R^k$. Let $f(x) = L(x)+F(x)$, where L is a linear function for which the plane $P(\langle x_0, f(x_0)\rangle)$ is the graph. By direct computation, one can show that differential $DF(x_0)$ is equal to 0, hence does exist, and so f, as a sum of a linear function L and of a differentiable function F, must be differentiable at x_0.

THEOREM. *Topological manifold M^k in a Euclidean space R^n is a C^r submanifold $(1 \leqslant r \leqslant \infty)$ if and only if*

1. *for each $x \in M^k$, M^k has a k-dimensional tangent plane $P(x)$,*

2. *if $P_0(x)$ is the k-dimensional linear subspace of R^n parallel to $P(x)$, then the map*

$$P_0\colon M^k \to G_{n,k}\colon x \mapsto P_0(x)$$

is of class C^{r-1},

3. *for each $x \in M^k$, the orthogonal projection π_x: $M^k \to P(x)$ is a homeomorphism on some neighbourhood of x.*

Proof. Necessity. Let M^k be a C^r submanifold of R^n. Take a chart (U, φ) at a point $x \in M^k$. Vectors $a_i(y) = [D_i\varphi^{-1}](\varphi(y))$, where $i = 1, 2, \ldots, k$, are tangent to M^k at y and span the plane $P_0(y)$. Hence condition 1 holds.

The map f: $U \to V^*_{n,k}$: $y \mapsto \langle a_1(y), \ldots, a_k(y)\rangle$ is obviously of the class C^{r-1}. If π: $V^*_{n,k} \to G_{n,k}$ is the natural projection, then $P_0|_U = \pi \circ f$ and, by Lemma 2, π is of class C^∞. Hence P_0 is of class C^{r-1}, and so condition 2 holds too.

Condition 3 is an easy conclusion from the inverse function theorem. In fact, $[D(\pi_x \circ \varphi^{-1})](\varphi(x))$ is non-degenerate ($D\pi_x$ is the projection onto $P_0(x)$), hence $\pi_x \circ \varphi^{-1}$ is reversible on some neighbourhood U of x, and

$$\pi_x|_U = (\pi_x \circ \varphi^{-1})|_{\varphi(U)} \circ (\varphi|_U)$$

is a homeomorphism.

Sufficiency. To proof the sufficiency one must show that, for each $x \in M$, π_x^{-1} considered on some neighbourhood U of x is a C^r immersion, because then $(U, \pi_x|_U)$ is a C^r atlas on M.

Let $x = (0, \ldots, 0)$, $P(x)$ be a subspace of R^n such that $x_{k+1} = x_{k+2} = \ldots = x_n = 0$, and U be a neighbourhood of x such that (i) π_x on U is a homeomorphism, and (ii) for each $y \in U$, π_x carries $P(y)$ onto $P(x)$.

Apply Lemma 3 for $p = P(x)$ and v^0 equal to the canonical base $\langle e_1, \ldots, e_k\rangle$ of $P(x)$. Let w and W be as in Lemma 3. Taking, if necessary, common part $U \cap P_0^{-1}(W)$, we can assume that $U \subset P_0^{-1}(W)$. Let $\varphi\colon \pi_x(U) \to V^*_{n,k}$ be the map defined by $\varphi = w \circ P_0 \circ (\pi_x|_U)^{-1}$.

Clearly, the class of φ is the minimum of $r-1$ and of the class of π_x^{-1}.

By Lemma 3,

$$\varphi(y) = \langle \varphi_1(y), \ldots, \varphi_k(y) \rangle = \langle e_1 + F_1(y), \ldots, e_k + F_k(y) \rangle,$$

where $F_i(y) \in (P(x))^\perp$, $\varphi_i(y) \in P_0(y)$, $\pi_x(\varphi(y)) = e_i$.

The differential $D_i \pi_x^{-1}$ can be written in the form

$$[D_i \pi_x^{-1}](y) = e_i + G_i(y),$$

where $G_i(y) \in (P(x))^\perp$, $[D_i \pi_x^{-1}](y) \in P_0(y)$, $\pi_x([D_i \pi_x^{-1}](y)) = e_i$.

Hence $\varphi_i(y) = D_i \pi_x^{-1}(y)$ for $y \in U$, because π_x is 1-1 on $P_0(y)$. Therefore, π_x^{-1} is a solution of the system of differential equations

$$D_i X = \varphi_i. \tag{1}$$

In view of (i), π_x^{-1} is continuous, i. e. of the class C^0. Suppose that π_x^{-1} is of a class C^s, where $0 \leqslant s < r$. Then each function φ_i is of the same class C^s and so π_x^{-1} is, as a solution of system (1), of the class C^{s+1}. Hence, by easy induction, π_x^{-1} must be of the class C^r.

Since φ_i does not vanish, π_x^{-1} is a C^r immersion. The proof is complete.

Remarks. The first condition in theorem is obviously essential and easy examples show that the two other are such. For instance, if we take the manifold

$$M = \left\{(x, y)\colon\ y = x^2 \sin\frac{1}{x} \text{ for } x \neq 0 \text{ or } y = x = 0\right\},$$

then the tangent line $P(p)$ does exist at any point $p \in M$ and the orthogonal mapping $\pi_p\colon M \to P(p)$ is a homeomorphism on a certain neighbourhood U of p, but condition 2 fails: the map $P_0\colon M \to G_{2,1}$ is not even continuous at the point $(0, 0)$. And if we consider the manifold $M = \{(x, y)\colon y^3 = x^2\}$, then the tangent line $P(p)$ also does exist for any point $p \in M$ and the mapping $P_0\colon M \to G_{2,1}$ is continuous, but this time condition 3 fails: for the point $p_0 = (0, 0)$ the projection $\pi_{p_0}\colon M \to P(p_0)$ is not a homeomorphism on any neighbourhood of p_0. In both cases, however, M is a submanifold of R^2 of class C^0 only, i. e., a topological but not a differentiable one.

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