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ON LOCALLY FINITE COVERINGS

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1. Introduction. It is well known that the notions of a locally finite open covering and of a σ-discrete open covering are tightly related. For example, the statements that every open covering of a regular space admits a locally finite (respectively, σ-discrete) open refinement are equivalent and both characterize paracompact spaces. Let us consider the following two properties of topological spaces:

(A) Every σ-discrete open covering admits a locally finite open refinement.

(B) Every locally finite open covering admits a σ-discrete open refinement.

One easily observes that property (A) is in fact equivalent to countable paracompactness. Therefore – in virtue of Rudin's example [6] of a normal non-countably paracompact space – there exist normal spaces that do not satisfy (A).

On the other hand, it is known that each normal space satisfies (B) (see [2], Exercise 5.1.I). G. M. Reed asks if every regular space satisfies (B). The following example gives a negative answer to his question.

Example 1. A metacompact Moore (1) space X and its locally finite open covering $\mathscr{V}$ with no σ-discrete open refinement.

The covering $\mathscr{V}$ has cardinality ω_1 (the smallest possible) and is locally of order not greater than 3. Example 1 is constructed in Section 2.

In 1958 Katětov [4] raised the following still open problem:

Problem (Katětov). Can each locally finite open covering of a closed subspace F of a collectionwise normal space X be extended to a locally finite open covering of X? (See the note Added in Proof.)

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(1) A regular space X is a *Moore space* if there exists a family $\{\mathscr{A}_n\}_{n\in\omega}$ of open coverings of X such that for each point $x \in X$ and its neighborhood U there exists an $n \in \omega$ with $x \in \mathrm{St}(x, \mathscr{A}_n) = \bigcup\{A \in \mathscr{A}_n : x \in A\} \subset U$.

Katětov obtained the following partial results:

Definition. A covering $\mathscr{A}$ is *uniformly locally finite* if there exists a locally finite open covering $\mathscr{B}$ such that each $B \in \mathscr{B}$ intersects only finitely many elements of $\mathscr{A}$.

FACT 1. *Every uniformly locally finite open covering of a closed subspace F of a collectionwise normal space can be extended to a uniformly locally finite open covering of X.*

FACT 2. *Every locally finite covering of a collectionwise normal countably paracompact space is uniformly locally finite.*

In connection with these results a natural question arises: Is every locally finite open covering of a collectionwise normal space uniformly locally finite? The following example answers this question in the negative.

Example 2. A collectionwise normal space Y and its (countable, cozero) locally finite open cover which is not uniformly locally finite.

In [5] the notion of a weak cb-space was defined. A normal space is a *weak cb-space* if every countable covering consisting of regularly open ([2]) sets admits a locally finite open refinement.

Therefore, when restricted to the class of normal spaces, this notion is a weak form of countable paracompactness. Mack and Johnson [5] ask (see also [1], p. 249) if there exists a normal space which is not a weak cb-space. It turns out that Example 2 provides a negative answer. (The last question has been answered independently by Hardy and Juhàsz [3].)

Example 2 is constructed in Section 3.

2. Construction of Example 1. The following, supposedly known, lemma has been communicated to the author by Eric K. van Douwen.

LEMMA. *There exists a family $\{A_a\}_{a<2^{\omega_0}}$ of disjoint subsets of the real line $\boldsymbol{R}$ such that for each $a < 2^{\omega_0}$ and a non-empty open subset U of $\boldsymbol{R}$ the set $A_a \cap U$ is of second category in $\boldsymbol{R}$, i.e. $A_a \cap U$ is not the union of a countable sequence of nowhere dense subsets of $\boldsymbol{R}$.*

Proof. Let us enumerate by $\{D_\beta\}_{\beta<2^{\omega_0}}$ all uncountable G_δ-subsets of $\boldsymbol{R}$ in such a way that each uncountable G_δ-subset appears in that sequence continuum many times. Let us also recall that every uncountable G_δ-subset of $\boldsymbol{R}$ has cardinality continuum (see [2], Exercise 4.5.5). For each a, $\beta < 2^{\omega_0}$ with $a \leqslant \beta$ choose a point $x_{\beta,a}$ in such a way that

$$x_{\beta,a} \in D_\beta \setminus \{x_{\gamma,\delta}\colon\ \gamma < \beta \text{ or } \gamma = \beta \text{ and } \delta < a\}.$$

([2]) A subset U of X is *regularly open* if $U = \operatorname{Int}\overline{U}$.

Put $A_a = \{x_{\beta,a}\colon a \leqslant \beta\}$ and observe that the sets A_a are disjoint and for each $\beta < 2^{\omega_0}$ we have $|A_a \cap D_\beta| = 2^{\omega_0}$. Let U be an arbitrary non-empty open subset of $\boldsymbol{R}$. If $A_a \cap U$ were of first category in $\boldsymbol{R}$, then there would exist a sequence $\{F_n\}_{n\in\omega}$ of closed and nowhere dense subsets of $\boldsymbol{R}$ such that

$$A_a \cap U \subset \bigcup_{n\in\omega} F_n.$$

This is however impossible, since $U \setminus \bigcup_{n\in\omega} F_n$ is an uncountable G_δ-subset of $\boldsymbol{R}$ and, therefore,

$$A_a \cap \Big(U \setminus \bigcup_{n\in\omega} F_n\Big) \neq \emptyset.$$

We shall define a new topology on the plane $\boldsymbol{R}^2$. Let $\{A_a\}_{a<2^{\omega_0}}$ be a family of subsets of $\boldsymbol{R}$ satisfying conditions of Lemma 1 and let $\{B_n\}_{n<\omega}$ be a countable base for the real line consisting of non-empty sets. Since for each $n < \omega$ the set $A_n \cap B_n$ is uncountable, we can choose a family $\{x^n_{a,\beta}\}_{a<\beta<\omega_1}$ of its distinct points.

In the new topology on $\boldsymbol{R}^2$, all points $(x, y) \in \boldsymbol{R}^2$ such that $y \neq 0$ will be isolated and the base $\{U_m(x)\}_{m<\omega}$ of neighborhoods of the point $(x, 0)$ will be defined in the following way:

1° if there exist $n \in \omega$ and $a < \beta < \omega_1$ such that $x = x^n_{a,\beta}$, then

$$U_m(x) = \left\{(t, v) \in \boldsymbol{R}^2\colon\ t = v + x,\ t \in A_a \cup A_\beta \cup \{x\} \text{ and } 0 \leqslant v < \frac{1}{m+1}\right\};$$

2° otherwise,

$$U_m(x) = \left\{(x, v) \in \boldsymbol{R}^2\colon\ 0 \leqslant v < \frac{1}{m+1}\right\}.$$

Denote by X the set $\boldsymbol{R}^2$ with the above-defined topology. One easily checks that X is a completely regular, metacompact Moore space.

For each a such that $\omega \leqslant a \leqslant \omega_1$ define open subsets V_a of X by putting

$$V_a = \{(x, y) \in \boldsymbol{R}^2\colon\ x \in A_a\} \quad \text{if } \omega \leqslant a < \omega_1$$

and

$$V_{\omega_1} = X \setminus \{(x, 0) \in \boldsymbol{R}^2\colon\ x \in A_a \text{ and } \omega \leqslant a < \omega_1\}.$$

Clearly, the family $\mathscr{V} = \{V_a\}_{\omega\leqslant a\leqslant\omega_1}$ forms an open covering of the space X and each point of the space X has a neighborhood intersecting at most three elements of the family $\mathscr{V}$. Therefore, $\mathscr{V}$ is locally finite. We shall show that $\mathscr{V}$ does not have a σ-discrete open refinement.

For assume otherwise and let

$$\mathscr{G} = \bigcup_{k<\omega} \mathscr{G}_k$$

be an open refinement of $\mathscr{V}$ such that the family $\mathscr{G}_k$ is discrete for each $k < \omega$. For every k, $m < \omega$ and $\omega \leqslant a < \omega_1$ write

$$A_{a,k,m} = \{x \in A_a\colon \text{there exists a } G \in \mathscr{G}_k \text{ such that } U_m(x) \subset G \subset V_a\}.$$

Clearly, for each a such that $\omega \leqslant a < \omega_1$ we have

$$A_a = \bigcup_{k,m<\omega} A_{a,k,m}.$$

Therefore, since A_a's are not of first category in $\boldsymbol{R}$, for each $\omega \leqslant a < \omega_1$ there exist $k(a)$, $m(a)$ and $n(a)$ belonging to ω such that

$$B_{n(a)} \subset \bar{A}^{\boldsymbol{R}}_{a,k(a),m(a)}.$$

Choose two distinct ordinal numbers a and β such that $\omega \leqslant a < \beta < \omega_1$, and $k(a) = k(\beta) = k$, $m(a) = m(\beta) = m$ and $n(a) = n(\beta) = n$. We shall show that each neighborhood of the point $z = (x, 0)$, where $x = x^n_{a,\beta}$, intersects at least two elements of $\mathscr{G}_k$, which will contradict the discreteness of the family $\mathscr{G}_k$. Let us choose an arbitrary $l < \omega$ and consider the neighborhood $U_l(x)$ of the point z.

Since $x \in \bar{A}^{\boldsymbol{R}}_{a,k,m} \cap \bar{A}^{\boldsymbol{R}}_{\beta,k,m}$, there exist $t_1 \in A_{a,k,m}$ and $t_2 \in A_{\beta,k,m}$ such that

$$|t_1 - x| + |t_2 - x| < \frac{1}{l+m+2}.$$

By the definition of $U_l(x)$, we have

$$U_l(x) \cap U_m(t_1) \neq \emptyset \neq U_l(x) \cap U_m(t_2)$$

and there exist G_1, $G_2 \in \mathscr{G}_k$ such that

$$U_m(t_1) \subset G_1 \subset V_a \quad \text{and} \quad U_m(t_2) \subset G_2 \subset V_\beta.$$

The sets G_1 and G_2 are different since $V_a \cap V_\beta = \emptyset$, and we have

$$U_l(x) \cap G_1 \neq \emptyset \neq U_l(x) \cap G_2.$$

Therefore, we showed that an arbitrary neighborhood of z intersects two different elements of $\mathscr{G}_k$, which is a contradiction and completes the proof of the properties of the space X.

Remark. If in our construction we used rather all sets A_a, $a < 2^{\omega_0}$, instead of using only ω_1 of them, we would obtain an open locally finite covering of X of cardinality 2^{ω_0} having no open refinement consisting of less than 2^{ω_0} discrete families.

3. Construction of Example 2. Let X be an arbitrary example of a collectionwise normal space which is not countably paracompact, e.g. Rudin's example [6]. Hence there must exist an increasing sequence $\{U_n\}_{n<\omega}$ of open subsets of X, covering X and having no locally finite open refinement (see [2], Theorem 5.2.1). Let $Z^* = X \times \omega$ be the Cartesian product of X and of the space ω of natural numbers with the product topology, and let

$$Z = \bigcup_{n<\omega} (\bar{U}_n \times \{n\})$$

be the subspace of Z^*. Since Z is a closed subset of a collectionwise normal space Z^*, it is also collectionwise normal. Clearly, the increasing open covering $\{V_n\}_{n<\omega}$ of Z^*, where $V_n = Z \cap (U_n \times \{0, 1, 2, \ldots, n\})$, also does not have a locally finite open refinement. Moreover, since

$$Z \setminus \bar{V}_n = \bigcup_{k>n} (\bar{U}_k \times \{k\}),$$

the sets $Z \setminus \bar{V}_n$ are open F_σ-subsets of Z.

Let us define a modified topology on the set $Y = Z \times [0, 1]$. All points $(z, t) \in Y$ with $t > 0$ are isolated, and points of the form $(z, 0)$ have the same base of neighborhoods as in the product topology of Z and $[0, 1]$. One easily checks that the space Y (considered with the new topology) is collectionwise normal.

Let us consider an increasing countable open cover $\mathscr{G} = \{G_n\}_{n<\omega}$ of Y, where $G_n = V_n \times [0, 1]$.

Obviously, $\mathscr{G}$ does not have a locally finite open refinement in Y. Moreover, the sets G_n are regularly open in Y. Indeed,

$$\bar{G}_n^Y = (\bar{V}_n^Z \times \{0\}) \cup G_n$$

and, therefore, $\mathrm{Int}\bar{G}_n^Y = G_n$. This shows that Y is an example of a collectionwise normal space, which is not a weak cb-space.

Let us put $W_n = Y \setminus \bar{G}_n$. The sets

$$W_n = \big((Z \setminus \bar{V}_n) \times \{0\}\big) \cup (Z \setminus V_n) \times [0, 1]$$

are open F_σ-subsets of Y, and since Y is normal, they are cozero sets. Let us consider a countable cozero covering $\mathscr{W} = \{W_n\}_{n<\omega} \cup \{Y\}$ of Y. Clearly, $\mathscr{W}$ is locally finite, since $\mathscr{G}$ is an increasing covering of Y. It suffices to show that $\mathscr{W}$ is not a uniformly locally finite covering of Y. Assume that there exists a locally finite open covering $\mathscr{B}$ of Y such that each $B \in \mathscr{B}$ intersects only finitely many elements of $\mathscr{W}$ and for each $B \in \mathscr{B}$ find an $n(B) \in \omega$ such that $B \cap W_{n(B)} = \emptyset$.

Write $B_n = \{B \in \mathscr{B}: n(B) = n\}$ and observe that $\{B_n\}_{n<\omega}$ is a locally finite open covering of Y and that $B_n \cap W_n = \emptyset$. Therefore,

$$B_n \subset \mathrm{Int}(Y \setminus W_n) = \mathrm{Int}\bar{G}_n = G_n,$$

which is a contradiction, since $\mathscr{G}$ does not admit a locally finite open refinement.

Added in proof. The problem of Katětov, mentioned in the introduction, has been solved in the negative (see T. Przymusiński and M. Wage, *Collectionwise normality and extensions of locally finite coverings*, to appear in Fundamenta Mathematicae).

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