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ON PROBLEMS OF B. CHOCZEWSKI AND M. KUCZMA CONCERNING AN INTEGRAL EQUATION

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Lipiński [3] and subsequently Choczewski and Kuczma [1] considered the integral

$$F(t) = \int_a^b f[t\varphi(u)]\,du \quad \text{for } 0 \leqslant t \leqslant 1, \tag{1}$$

where $\varphi(u)$ is an increasing function on $\langle a, b\rangle$ such that $\varphi(a) = 0, \varphi(b) = 1$, and $f(x)$ is a real-valued continuous function on $\langle 0, 1\rangle$. Lipiński, answering the problem of Gołąb [2], showed that there exists a continuous increasing function φ such that the equality

$$F(t) = 0 \quad \text{for } 0 \leqslant t \leqslant 1 \tag{2}$$

does not imply the equality

$$f(x) = 0 \quad \text{for } 0 \leqslant x \leqslant 1. \tag{3}$$

Choczewski and Kuczma generalized the result of Lipiński and proved that for every positive integer r there exist a continuous strictly increasing function φ from $\langle a, b\rangle$ onto $\langle 0, 1\rangle$ and a non-trivial function f of class C^r on $\langle 0, 1\rangle$ such that for function (1) equality (2) holds. They also asked two new questions:

(P 791) Do there exist a continuous strictly increasing function φ from $\langle a, b\rangle$ onto $\langle 0, 1\rangle$ and a non-trivial function f of class C^∞ on $\langle 0, 1\rangle$ such that equality (2) holds?

(P 792) Do there exist a continuous strictly increasing function φ from $\langle a, b\rangle$ onto $\langle 0, 1\rangle$ and a non-trivial function on $\langle 0, 1\rangle$ having, for every positive integer r, the asymptotic property

$$f(x) = 0(x^r) \quad \text{for } x \to 0+0$$

and such that for function (1) equality (2) holds?

Generalizing the original construction given by Lipiński and modified by Choczewski and Kuczma we give a positive answer for these problems.

Put

$$x_n = 2^{-n}, \quad y_n = \frac{2n+3}{n+2} x_{n+1} \quad \text{for } n = 0, 1, 2, \ldots \tag{4}$$

Define a function $\varphi(u)$ on $\langle 0, 1\rangle$ by the conditions

$$\varphi(x_n) = x_{2n}, \quad \varphi(y_n) = x_{2n+1} \quad \text{for } n = 0, 1, 2, \ldots \tag{5}$$

and by the requirement that φ is linear in any intervals $\langle x_{n+1}, y_n\rangle$ and $\langle y_n, x_n\rangle$; moreover, $\varphi(0) = 0$. Thus

$$\varphi(u) = \begin{cases} x_{2n+2} + a_n(u - x_{n+1}) & \text{for } u \in \langle x_{n+1}, y_n\rangle, \\ x_{2n+1} + b_n(u - y_n) & \text{for } u \in \langle y_n, x_n\rangle, \end{cases} \tag{6}$$

where $n = 0, 1, 2, \ldots$ and

$$a_n = \left(1 + \frac{1}{n+1}\right) x_{n+1}, \quad b_n = (n+2)x_n. \tag{7}$$

It is obvious from the construction that φ is a continuous and strictly increasing map of $\langle 0, 1\rangle$ into itself, and that $\varphi(0) = 0, \varphi(1) = 1$.

LEMMA 1. *If $g(x)$ is a continuous function defined in the interval $\langle \frac{1}{2}, 1\rangle$ and satisfying the condition $g(\frac{1}{2}) = g(1) = 0$, then the function $f(x)$ defined by the recurrent formula*

$$f(x) = \begin{cases} g(x) & \textit{for } x \in \langle x_1, x_0\rangle, \\ -\dfrac{1}{n+1} f(2x) & \textit{for } x \in \langle x_{n+1}, x_n\rangle \textit{ and } f(0) = 0, \end{cases} \tag{8}$$

where $n = 1, 2, 3, \ldots,$ is a continuous extension of the function $g(x)$ onto the interval $\langle 0, 1\rangle$.

Proof. Since for $x \in \langle x_{n+1}, x_n\rangle$ we have

$$f(x) = (-1)^n \frac{1}{(n+1)!} g(2^n x), \tag{9}$$

and the continuity of the function $f(x)$ is obvious.

LEMMA 2. *If $f(x)$ and $\varphi(u)$ are defined as previously, then*

$$\int_0^1 f[t\varphi(u)]\,du = 0 \quad \textit{for } t \in \langle 0, 1\rangle. \tag{10}$$

Proof. By (8) we have $f(0) = 0$, i.e. (10) holds for $t = 0$. For $t \in (0, 1\rangle$ we first evaluate the integral

$$\int_{x_{n+1}}^{x_n} f[t\varphi(u)]\,du = \int_{x_{n+1}}^{y_n} f[t\varphi(u)]\,du + \int_{y_n}^{x_n} f[t\varphi(u)]\,du.$$

By (6) and (5),

$$\int_{x_{n+1}}^{y_n} f[t\varphi(u)]\,du = \int_{tx_{2n+2}}^{tx_{2n+1}} (ta_n)^{-1} f(x)\,dx$$

and, by (6), (5), and (8),

$$\int_{y_n}^{x_n} f[t\varphi(u)]\,du = \int_{tx_{2n+1}}^{tx_{2n}} (tb_n)^{-1} f(s)\,ds = \int_{tx_{2n+2}}^{tx_{2n+1}} 2(tb_n)^{-1} f(2x)\,dx$$
$$= -\int_{tx_{2n+2}}^{tx_{2n+1}} 2(n+1)(tb_n)^{-1} f(x)\,dx.$$

Hence

$$\int_{x_{n+1}}^{x_n} f[t\varphi(u)]\,du = t^{-1}[a_n^{-1} - 2(n+1)b_n^{-1}] \int_{tx_{2n+2}}^{tx_{2n+1}} f(x)\,dx = 0,$$

since in view of (7) and (4) the equality $[a_n^{-1} - 2(n+1)b_n^{-1}] = 0$ holds Thus also

$$\int_0^1 f[t\varphi(u)]\,du = \sum_{n=0}^{\infty} \int_{x_{n+1}}^{x_n} f[t\varphi(u)]\,du = 0 \qquad \text{for } t \in \langle 0,1\rangle,$$

i.e. relation (10) holds true.

THEOREM 1. *There are a continuous strictly increasing function φ from $\langle 0,1\rangle$ onto $\langle 0,1\rangle$ and a non-trivial function on $\langle 0,1\rangle$ having, for every positive integer r, the asymptotic property*

$$f(x) = 0(x^r) \quad \textit{for } x \to 0+0$$

and such that (10) *holds.*

Proof. If $x \in \langle x_{n+1}, x_n\rangle$, we have the following inequality:

$$\frac{-|f(x)|}{(x_{n+1})^r} \leqslant \frac{f(x)}{x^r} \leqslant \frac{|f(x)|}{(x_{n+1})^r}.$$

For every function $f(x)$ satisfying (8) equality (10) holds and by (9) we have

$$\frac{|f(x)|}{(x_{n+1})^r} = \frac{(2^r)^{n+1}}{(n+1)!}|g(2^n x)|.$$

Hence

$$\lim_{x\to 0+0} \frac{f(x)}{x^r} = 0$$

for every positive integer r.

Theorem 1 yields a positive answer to P 792.

LEMMA 3. *If $g(x)$ is a function of class C^∞, defined in the interval $\langle \frac{1}{2}, 1\rangle$ and satisfying the conditions*

$$g^{(n)}(\tfrac{1}{2}) = g^{(n)}(1) = 0 \quad \textit{for } n = 0, 1, 2, \ldots, \tag{11}$$

then the extension $f(x)$ defined by (8) *is a function of class C^∞ on $\langle 0, 1\rangle$.*

Assumptions (11) are satisfied, e.g., by the function

$$g(x) = \begin{cases} \exp\left(\dfrac{-1}{(x-\frac{1}{2})(1-x)}\right) & \text{for } x \in (\frac{1}{2}, 1), \\ 0 & \text{for } x = \frac{1}{2} \text{ and } x = 1. \end{cases}$$

Proof. Let $f_k(x)$ be an extension of the function $g^{(k)}(x)$ onto $\langle 0, 1\rangle$ for $k = 1, 2, 3, \ldots$ defined by the recurrent formula

$$f_k(x) = \frac{-2^k}{n+1} f_k(2x) \quad \text{for } x \in \langle x_{n+1}, x_n\rangle,$$

$$f_k(0) = 0.$$

Then, for $x \in \langle x_{n+1}, x_n\rangle$,

$$f_k(x) = (-1)^n \frac{2^{kn}}{(n+1)!} g^{(k)}(2^n x).$$

By (9) we have $f^{(k)}(x) = f_k(x)$ for $x \in (0, 1\rangle$. If $x = 0$, then

$$f^{(k)}(0) = \lim_{x \to 0+0} \frac{f^{(k-1)}(x)}{x} = 0 \quad \text{for } k = 1, 2, 3, \ldots$$

Hence functions $f_k(x)$ for $k = 1, 2, 3, \ldots$ are derivatives of the function $f(x)$.

From Lemmas 2 and 3 we obtain the following result which yields a positive answer to P 791.

THEOREM 2. *There is a continuous strictly increasing function φ from $\langle 0, 1\rangle$ onto itself and there are infinitely many non-trivial functions f of class C^∞ on $\langle 0, 1\rangle$ such that equality* (10) *holds.*

REFERENCES

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