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CHARACTERIZING DISCRETE VECTOR LATTICES

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Introduction. Conditions on a vector lattice which either imply or are equivalent to its being a discrete vector lattice have been of interest. In [6] Nakano characterized all Dedekind complete, discrete vector lattices. Zaanen in [9] obtained a necessary and sufficient condition for a Dedekind σ-complete vector lattice in which every principal band has a strong unit to be discrete and finite-dimensional. In [2] Komura and Koshi showed that every nuclear vector lattice is discrete. It is the purpose of this paper to obtain two characterizations of discrete vector lattices. One uses lattice properties only and the other combines lattice properties with topological considerations.

We shall prove* that an Archimedean vector lattice E is discrete if and only if it has sufficiently many projections and every f in E^+ (the positive elements of E) can be decomposed into components $\{f_a\colon a \in A\}$ such that if g is a component of f, then $\inf(g, f_a) = 0$ or $\inf(g, f_a) = f_a$ for all $a \in A$. The conditions assumed in this result combined with order separability imply the Egoroff property ([8], Theorem 2.3, p. 34). Since a vector lattice which is regular (the condition of [9], Theorem 6.3) also has the Egoroff property (see Section 1 of Chapter 1 in [8] or [3], Chapter 10), we can see a relationship between our result and Zaanen's result.

In the last section we shall show that the existence of a collection $\{\|\cdot\|_a\colon a \in A\}$ of compatible continuous seminorms defining a locally convex Hausdorff topology on a universally complete vector lattice is necessary and sufficient for it to be discrete.

1. Preliminaries. E will represent an Archimedean vector lattice throughout this paper.

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The element $f \neq 0$ in E is called an *atom* or a *discrete element* whenever it follows from $0 \leqslant u \leqslant |f|$, $0 \leqslant v \leqslant |f|$ and $\inf(u, v) = 0$ that $u = 0$ or $v = 0$.

It is known ([9], Theorem 4.1) that if f is an atom and $0 \leqslant u \leqslant |f|$, then $u = af$, where a is a real number.

A set $\{f_a: a \in A\}$ of positive elements *generates* E if the elements are mutally disjoint ($\inf(f_a, f_\beta) = 0$ if $a \neq \beta$) and $\inf(f_a, x) = 0$ for all $a \in A$ implies that $x = 0$.

A *discrete vector lattice* is an Archimedean vector lattice with a generating set of discrete elements.

Note that we do not assume that a discrete space is Dedekind complete as in [6]. It is true that E is discrete if and only if its Dedekind completion is discrete and the cardinality of the generating sets is the same (see [5], Theorem 3.3, p. 477). In fact, one can use this latter statement to remove Nakano's restriction of Dedekind completeness and his characterizations remain valid.

In the sequel, we shall denote a principal band in E generated by an element x by $B(x)$. Given any elements y and x in E^+, we write $[x]y$ for the projection of y onto $B(x)$ whenever this projection exists. Recall that

$$[x]y = \sup\{\inf(y, nx): n \in N\} = \sup\{z: 0 \leqslant z \leqslant y, z \in B(x)\}$$

whenever this supremum exists. A principal band $B(x)$ in E is a *projection band* if $[x]y$ exists for all y in E^+. We refer the reader to [9], p. 163-165, for a discussion of projections. A vector lattice is said to be of *strong countable type* if every set of mutually disjoint elements is at most countable.

Any of the usual sequence spaces (linear subspaces of the space of all real sequences which contain the finitely non-zero elements and are Archimedean vector lattices) are examples of discrete vector lattices. In fact, we have the following

THEOREM 1.1. *A discrete vector lattice is isomorphic to a sequence space or a finite-dimensional space if and only if it is of strong countable type.*

Proof. Since every sequence space contains a countable collection of discrete elements which generates the space, it is of strong countable type. The reasoning is similar for a finite-dimensional space where the above-mentioned collection is finite.

Now suppose that E is discrete and of strong countable type. Let $\{p_a: a \in A\}$ be a generating set of discrete positive elements in E. Then $\{p_a: a \in A\}$ must be countable and we can re-index the set to $\{p_n: n \in P\}$ where P is a subset of the positive integers. If P is finite, then we immediately get that E is finite-dimensional ([9], Theorem 4.2). So, we may assume that P is the set of all positive integers. Since a band generated by an atom is always a projection band ([9], Theorem 4.1), we know that, for every

f in E^+, $[p_n]f$ exists for all n in P. It also follows that $[p_n]f = a_n p_n$, where a_n is a real number. Define the function $\bar{f}$ on P by $\bar{f}(n) = a_n$ for each n in P. $\bar{f}$ is a real sequence and is defined uniquely for each f in E^+. If f is any element in E (not necessarily positive), we can define $\bar{f} = \overline{f^+} - \overline{f^-}$. Let $M = \{\bar{f}$: $\bar{f}$ corresponds to each f in E as described$\}$. M is a sequence space and E is isomorphic to M. The theorem is established.

Therefore, in the presence of strong countability, one can think of a discrete vector lattice as a sequence space or a finite-dimensional space. One can also think of a discrete space as a vector lattice of measurable functions over a completely atomic measure space as long as the characteristic function of each of the atoms in the Boolean algebra of measurable sets is included. Such a space of functions would be isomorphic to a sequence space if and only if the measure space is σ-finite. Hence, in the theorems in the following sections, the reader should keep in mind the corresponding characterizations for sequence spaces and spaces of measurable functions over a completely atomic measure space.

2. First characterization. The element f in a vector lattice E is a *component* of g in E^+ if and only if $\inf(f, g-f) = 0$. E is said to have *property* (P) if for every f in E^+ there is a collection $\{f_a\colon a \in A\}$ of components of f such that $\sup\{f_a\colon a \in A\} = f$ and if g is a component of f, then $\inf(f_a, g) = 0$ or $\inf(f_a, g) = f_a$ for all a in A.

THEOREM 2.1. *An Archimedean vector lattice E is discrete if and only if E satisfies* (P) *and every non-zero principal band contains a non-zero principal projection band.*

We note that property (P), at first glance, appears to be quite strong and, in fact, appears to lead to the existence of atoms. However, $C[0,1]$, the continuous real-valued functions on $[0,1]$, satisfies (P) and does not contain any atoms. Before proving this statement and Theorem 2.1 we need some information concerning components of elements in a vector lattice.

LEMMA 2.1. *Let E be an Archimedean vector lattice and f, e_1, e_2 be elements of E^+.*

(i) *If e_1 and e_2 are components of f, then* $\sup(e_1, e_2)$ *is a component of f.*

(ii) *If $e_1 \leqslant e_2$ and e_1, e_2 are components of f, then e_1 is a component of e_2.*

(iii) *If e_1 and e_2 are components of f, then* $\inf(e_1, e_2)$ *is a component of f.*

A proof of this lemma can be found in [8], Remark 2.3, p. 31. As the proof is not difficult and follows from standard lattice identities which can be found in [3] or [7], we will not produce it here.

Example. $C[0,1]$ satisfies (P). Choose $f > 0$ in $C[0,1]$ and let $U = \{x\colon f(x) > 0\}$. U is open and is, therefore, the union of a countable collection F of disjoint open intervals. The intervals in F are of the form (a, b), $[0, b)$, $(a, 1]$ or $[0, 1]$, where $a, b \in [0, 1]$ and $a < b$. For simplicity we will assume that all the intervals in F are of the form (a, b). It will be clear from the proof for this situation how to handle the others.

Assume $F = \{(a_k, b_k): k = 1, 2, \ldots\}$ and for each k let $f_k = f\chi_k$, where χ_k is the characteristic function of (a_k, b_k). It is not difficult to see that each f_k is continuous and is a component of f. Also, we have $\sup\{f_k: k = 1, 2, \ldots\} = f$.

Suppose there is a component g of f and a positive integer k such that $\inf(g, f_k) \neq 0$ and $\inf(g, f_k) \neq f_k$. Letting $g' = \inf(g, f_k)$ we see that g' and $f_k - g'$ are non-zero components of f_k. (Apply Lemma 2.1, (iii) and (ii).) But then

$$\{x: g'(x) > 0\} \cap \{x: f_k(x) - g'(x) > 0\} = \emptyset$$

and

$$\{x: g'(x) > 0\} \cup \{x: f_k(x) - g'(x) > 0\} = (a_k, b_k).$$

This contradicts the connectedness of (a_k, b_k).

Proof of Theorem 2.1. Assume that E is a discrete vector lattice Let $B(u)$ be any non-zero principal band in E. Since E is discrete, there. is some atom p in E^+ such that $0 < p \leqslant |u|$. Since every band generated by an atom is a projection band, we infer that $B(p)$ is a projection band contained in $B(u)$. Now, let $f \in E^+$ and let $\{p_a: a \in A\}$ be a generating set of positive atoms in E. For each $a \in A$, let $f_a = [p_a]f$. It follows easily from the theory in Section 3 of [9] that each f_a is a component of f and that $\sup\{f_a: a \in A\} = f$. Furthermore, it can be shown using [9], Theorem 4.1, p. 166, that each non-zero f_a is an atom. Lemma 2.1 and what has been established above can now be used to see that if g is any component of f, then $\inf(g, f_a) = 0$ or $\inf(g, f_a) = f_a$ for all $a \in A$. We have established the necessity.

For the sufficiency, choose f in E^+ and let $\{f_a: a \in A\}$ be the collection mentioned in (P). We shall show that every f_a is an atom. Let $0 < u \leqslant f_a$. By assumption, there is a principal projection band $B(u') \subseteq B(u)$, where $u' > 0$. Also, $B(u') \subseteq B(f_a)$ since $u \leqslant f_a$. Now, $[u']f = v$ is a component of f and $v \in B(u') \subseteq B(u) \subseteq B(f_a)$ implies that $\inf(v, f_a) \neq 0$. But then using (P) we have $\inf(v, f_a) = f_a$. Hence $v \geqslant f_a$. But $v \geqslant f_a$, and v and f_a components of f imply that f_a is a component of v (Lemma 2.1 (ii)). Hence, $\inf(f_a, v - f_a) = 0$. But, $v, f_a \in B(f_a)$ implies that $v - f_a \in B(f_a)$. Then $\inf(v - f_a, f_a) = 0$ if and only if $v - f_a = 0$. Hence $v = f_a$. Then $f_a \in B(u)$ which implies that $B(u) = B(f_a)$. Now if $0 \leqslant u, v \leqslant f_a$ and $\inf(u, v) = 0$, we must have $u = 0$ or $v = 0$ since if $u \neq 0$ and $v \neq 0$, then $B(u) = B(v)$ which is impossible if $\inf(u, v) = 0$. Hence each f_a is an atom. Now let $\{p_\beta: \beta \in B\}$ be a maximal collection of mutually disjoint atoms in E. This collection is then a generating set of discrete elements in E and E is discrete.

A vector lattice E is said to be *order separable* if every subset A of E that has a supremum in E contains a countable subset A' such that $\sup A = \sup A'$. It is not difficult to see that if E satisfies the conditions of the previous theorem, then every f in E^+ can be decomposed into a countable number of components if and only if E is order separable. However, E need not be a sequence space as the example $E = \{f\colon f$ is a bounded real-valued function on an uncountable set X, where f is non-zero on a countable subset$\}$ illustrates. Here E is order separable and discrete but not of strong countable type and not a sequence space.

3. Second characterization. A vector lattice E is *universally complete* if it is Dedekind complete and $\sup\{x_a\colon a \in A\}$ exists in E whenever $\{x_a\colon a \in A\}$ is a collection of mutually disjoint elements in E. A seminorm $\|\cdot\|$ on E is said to be *compatible* if $\|x\| \leqslant \|y\|$ whenever $|x| \leqslant |y|$ and x and y are in E. $\|\cdot\|$ is said to be *continuous* if $\inf\{\|x_a\|\colon a \in A\} = 0$ whenever $0 \leqslant x_a \downarrow 0$ (i. e., $\inf\{x_a\colon a \in A\} = 0$ in E and $\{x_a\}$ is a net in E which is directed downward). We shall prove the following

THEOREM 3.1. *A universally complete vector lattice E is discrete if and only if there is a collection $\{\|\cdot\|_a\colon a \in A\}$ of compatible continuous seminorms defining a locally convex Hausdorff topology on E.*

Komura and Koshi ([2], Theorem 3) proved the related result that a locally convex vector lattice $E(T)$ (T a locally convex Hausdorff topology on E generated by compatible seminorms) is discrete whenever E is Dedekind complete and T is nuclear.

Before we prove Theorem 3.1 we shall introduce some terminology. A linear functional φ defined on E is said to be *order continuous* if $\inf\{\varphi(x_a)\colon a \in A\} = 0$ whenever $\{x_a\}$ is a net in E which converges down to 0. Let $E_n^{\sim}$ be the set of all order continuous linear functionals on E.

Proof of Theorem 3.1. If E is discrete and $\{p_a\colon a \in A\}$ is a generating set of positive atoms in E, then for each a in A and f in E^+ $[p_a]f = a_a p_a$, where a_a is a real number. Hence for each $a \in A$ and $f \in E^+$ we can define $\|f\|_a = a_a$. For an arbitrary f in E we define $\|f\|_a = \||f|\|_a$. Each $\|\cdot\|_a$ is well defined and $\{\|\cdot\|_a\colon a \in A\}$ is a collection of compatible continuous seminorms defining a locally convex Hausdorff topology on E.

On the other hand, assume E has the described collection of seminorms $\{\|\cdot\|_a\colon a \in A\}$. It is not difficult to show that E' (the topological dual) is contained in $E_n^{\sim}$ so that $E_n^{\sim}$ must be separating on E. Using a result of Masterson ([4], Corollary 3) it follows that E is isomorphic (algebraically and lattice) to a space $M(X, S, \mu)$ of equivalence classes of μ-measurable, almost everywhere finite-valued functions on a completely additive, localizable measure space (X, S, μ). We shall denote by S^* the Boolean algebra of equivalence classes of measurable subsets in S. Also, given any f in $M(X, S, \mu)$, we shall denote its carrier in S^* by f^*.

Let g be any non-zero element in $E = M(X, S, \mu)$. Under our assumptions on E we shall show that g^* must contain an atom. (An *atom* e^* in a Boolean algebra is a non-zero element such that whenever $e^* \supseteq y^* \supseteq 0^*$, then $y^* = e^*$ or $y^* = 0^*$.) Choose any f in E^+ with $0^* \neq f^* \subseteq g^*$ and $0 < \mu(f^*) < +\infty$. We shall show that f^* (and hence g^*) must contain an atom.

Assume f^* does not contain an atom. Then there exist elements $f^*(1)$, $f^*(2)$ in S^* such that $f^*(1) \cup f^*(2) = f^*$, $f^*(1) \cap f^*(2) = 0^*$ and $\mu(f^*(1)) = \mu(f^*(2)) = \frac{1}{2}\mu(f^*)$. We can continue this process so that for $n = 1, 2, \ldots$ and $i_n = 1$ or 2 we can choose $f^*(i_1, i_2, \ldots, i_n, 1)$ and $f^*(i_1, i_2, \ldots, i_n, 2)$ with

$$f^*(i_1, i_2, \ldots, i_n, 1) \cup f^*(i_1, i_2, \ldots, i_n, 2) = f^*(i_1, i_2, \ldots, i_n),$$
$$f^*(i_1, i_2, \ldots, i_n, 1) \cap f^*(i_1, i_2, \ldots, i_n, 2) = 0^*$$

and

$$\mu(f^*(i_1, i_2, \ldots, i_n, 1)) = \mu(f^*(i_1, i_2, \ldots, i_n, 2)) = \frac{1}{2}\mu(f^*(i_1, i_2, \ldots, i_n)) = \frac{1}{2^n}\mu(f^*).$$

Note that for any sequence $i_1, i_2, \ldots$ $(i_n = 1$ or $2)$ we have

$$f^*(i_1) \supset f^*(i_1, i_2) \supset \ldots \supset f^*(i_1, i_2, \ldots, i_n) \supset \ldots$$

Also,

$$\bigcap_{n=1}^{\infty} f^*(i_1, i_2, \ldots, i_n) = 0^*$$

since

$$\mu\Big(\bigcap_{n=1}^{\infty} f^*(i_1, i_2, \ldots, i_n)\Big) = \lim_{n\to\infty} \frac{1}{2^n}\mu(f^*) = 0.$$

Now for any sequence $i_1, i_2, \ldots$ $(i_n = 1$ or $2)$ let $f(i_1, i_2, \ldots, i_n)$ denote f multiplied by the characteristic function of $f^*(i_1, i_2, \ldots, i_n)$. Since $E_n^{\sim}$ is separating on E, there is a continuous linear functional φ with $\varphi(f) > 0$. Choose a specific subsequence $i_1, i_2, \ldots, i_n, \ldots$ $(i_n = 1$ or $2)$ such that $\varphi(f(i_1, i_2, \ldots, i_n)) > 0$ for each n. This can be done because $\varphi(f(1)) + \varphi(f(2)) \geqslant \varphi(f) > 0$ so that at least one of $\varphi(f(1))$ or $\varphi(f(2))$ is strictly positive.

Let $f_1 = f(i_1)$, $f_2 = f(i_1, i_2), \ldots,$ $f_n = f(i_1, i_2, \ldots, i_n), \ldots$ Then $f_1 \geqslant f_2 \geqslant \ldots,$ $\varphi(f_n) > 0$ for each n and $\inf\{f_n\colon n = 1, 2, \ldots\} = 0$. Hence $\inf\{\varphi(f_n)\colon n = 1, 2, \ldots\} = 0$. We can assume without loss of generality that $\varphi(f_n) \geqslant 2\varphi(f_{n+1})$ for $n = 1, 2, \ldots$ If not, we could choose a subsequence. Now for $n = 1, 2, \ldots,$ let

$$h_n = (n/\varphi(f_{n+1}))(f_n - f_{n+1}).$$

(Note the similarity here with [1], p. 119, Criterion 2.) $\{h_n\}$ is a collection of mutually disjoint elements in $M(X, S, \mu)$ and since M is universally complete, $\sup\{h_n: n = 1, 2, \ldots\} = h$ must exist in M. But

$$\varphi(h) \geqslant \varphi(h_n) = \big(n/\varphi(f_{n+1})\big)\big(\varphi(f_n) - \varphi(f_{n+1})\big) \geqslant 2n - n = n.$$

Hence, $\varphi(h) \geqslant n$ for each n, which is impossible. This contradiction proves that f^* and g^* must contain an atom.

Let $\{e^*_\beta: \beta \in B\}$ be a maximal collection of mutually disjoint atoms in S^*. For each β in B let p_β be the characteristic function of e^*_β. Then $\{p_\beta: \beta \in B\}$ forms a generating set of discrete elements for E.

COROLLARY 3.1. *A universally complete vector lattice E is the space of all real sequences or the space of all real n-tuples for some $n = 1, 2, \ldots$ if and only if there is a countable collection $\{\|\cdot\|_n: n \in N\}$ of compatible continuous seminorms defining a locally convex Hausdorff topology.*

Proof. We need only show that the existence of a countable collection of the described seminorms forces E to be order separable. This is true because in a universally complete vector lattice the concepts of order separability and strong countability coincide. Then, using Theorem 1.1 and the fact that the only universally complete sequence space is the space of all real sequences and the only universally complete n-dimensional space is the space of all real n-tuples, we have the desired result.

Let $\{x_a\}$ be any net in E with $0 \leqslant x_a \uparrow x$. If we can show that there is a sequence $\{x_{a(n)}\} \subseteq \{x_a\}$ such that $0 \leqslant x_{a(n)} \uparrow x$, then we would have E order separable. Since $\|\cdot\|_k$ is continuous for each k, we have

$$\inf\{\|x - x_a\|_k: a \in A\} = 0 \quad \text{for } k = 1, 2, \ldots$$

For $n = 1, 2, \ldots$, choose $x_{a(n)} \in \{x_a\}$ such that $x_{a(n)}\uparrow$ and $\|x - x_{a(n)}\|_k < 1/n$ for $k = 1, 2, \ldots, n$. Then for any $k \in N$ and any $\varepsilon > 0$ we can choose $K > 0$ such that $1/K < \varepsilon$ and $K \geqslant k$. Then

$$\|x - x_{a(n)}\|_k < 1/K < \varepsilon \quad \text{for all } n \geqslant K.$$

Hence

$$\inf\{\|x - x_{a(n)}\|_k: n \in N\} = 0 \quad \text{for } k = 1, 2, \ldots$$

But this forces $0 \leqslant x_{a(n)}\uparrow x$.

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