COLLOQUIUM MATHEMATICUM

VOL. LV | 1988 | FASC. 2

# THE MULTIPLIERS OF A SPACE OF ALMOST CONVERGENT CONTINUOUS FUNCTIONS

BY

F. BALIBREA (MURCIA)

**1. Introduction.** Let $l^\infty$ be the space of the real bounded sequences $x = (x(n))$. In [7] Lorentz introduced the *almost convergent real sequences* as those elements $x$ of $l^\infty$ for which all the Banach limits give a constant value, called the *almost limit* of $x$, and characterized them by the existence of the limit

$$\lim_n \Big(\sum_{i=k}^{k+n} x(i)\Big)/(n+1) \qquad \text{uniformly in } k \in N.$$

It is well known that a Banach limit $\mu$ on $l^\infty$ is an invariant mean related to the map $\sigma\colon N \to N$, $\sigma(n) = n+1$, that is an element $\mu$ of $(l^\infty)^*$ that satisfies $\mu(1, 1, \ldots) = 1$, $\mu(x) \geqslant 0$ if $x(n) \geqslant 0$ for every $n$, and $\mu(x \circ \sigma) = \mu(x)$ for every $x$ of $l^\infty$.

In [3], Theorem 3.1, Ching-Chou characterized the multipliers of the closed subspace $F$ of $l^\infty$ composed of the almost convergent sequences by proving that, given an $x \in l^\infty$, a necessary and sufficient condition for $xF \subset F$ is that $x$ be convergent along a filter basis $V$ composed of the subsets $M \subset N$ whose characteristic function is almost convergent to one (according to the Lorentz characterization, such subsets are the complements of the subsets of $N$ with uniform density zero). Later, Ching-Chou and Duran in [4] extended this result to the case of bounded functions defined on a left amenable countable semigroup, without a left finite ideal and satisfying the left cancellative law. Duran in [6] made the extension to the case of almost convergence with respect to more general subsets of means than the Banach limits.

The purpose of this paper is to extend the Ching-Chou result in another direction, substituting $N$ by a paracompact locally compact and non-compact topological space, and $\sigma$ by a proper continuous map $h\colon T \to T$ which satisfies the condition that for each compact $K \subset T$ there exists $n \in N$ such that $h^n(K) \cap K = \emptyset$. An example of such a situation is given in $N$ by a one-to-one map $\tau$ without a finite orbit.

The main result is the Theorem where we characterize the multipliers of the space of bounded continuous functions that are almost convergent with respect to the family $J$ of $h$-invariant means. The multipliers are precisely the functions convergent along a certain filter basis associated with $J$.

Parts of this paper have already been developed by the author in a more detailed and more general situation in [2].

**2. Notation and preliminary results.** Let $T$ be a completely regular Hausdorff topological space, and $V$ a filter basis in $T$. By $C_b(T)$ we denote the linear space of the real-valued bounded continuous functions defined on $T$ with the uniform norm. Let $Z(T)$ be the lattice composed of the subsets of $T$ that are zeros of continuous functions. The algebra of parts of $T$ generated by $Z(T)$ will be denoted by $A(T)$ (Baire algebra on $T$).

The dual of $C_b(T)$ can be identified by the Alexandrov Theorem [1] with the space $\mathcal{M}(T)$ composed of the finitely additive measures $\mu$, defined on $A(T)$, that are bounded and regular related to $Z(T)$ with the total variation norm $\|\mu\| = |\mu|(T)$. We denote by $\mathcal{M}_\tau(T)$ and $\mathcal{M}_t(T)$ the subspaces of $\mathcal{M}(T)$ composed of the $\tau$-additive and $t$-additive measures on $T$, respectively. A measure $\mu \in \mathcal{M}(T)$ is said to be *$\tau$-additive* if whenever a net $Z_\alpha$ of elements from $Z(T)$ is decreasing to the empty set ($Z_\alpha \searrow \emptyset$) it follows that $\mu(Z_\alpha) \to 0$. A measure $\mu \in \mathcal{M}(T)$ is said to be *t-additive* if for every $\varepsilon > 0$ there exists a compact $K \subset T$ such that

$$|\mu|_*(T\backslash K) = \sup\{|\mu|(Z)\colon\ T\backslash K \supset Z \in Z(T)\} < \varepsilon.$$

It can be proved that $\mathcal{M}_t(T) \subset \mathcal{M}_\tau(T)$. Moreover, the $\tau$-additive (resp., $t$-additive) measures can be extended in a unique way to $\sigma$-additive measures defined on the Borel $\sigma$-algebra of $T$ and internally regular related to the closed (resp., compact) subsets of $T$. In the following, we will suppose this extension has been made. In this way, $\mathcal{M}_t(T)$ will be identified with the space of finite Radon measures on $T$. When $T$ is locally compact or a complete metric space, it is well known that $\mathcal{M}_\tau(T) = \mathcal{M}_t(T)$ (see [11], Theorem 8.5).

In the following, we will denote by $c_0^V(T)$ the subspace of $C_b(T)$ composed of the functions that converge to zero along $V$, and by $c(T)$ the direct sum $c_0^V(T) \oplus \boldsymbol{R}$, that is, the subspace of $C_b(T)$ composed of the functions convergent along $V$. We will suppose that $c_0^V(T)$ is not empty (this is verified in the examples which will be considered).

We denote by $\mathcal{M}(T)_1^+$ the set

$$\{\mu \in \mathcal{M}(T)\colon\ \mu(f) \geqslant 0 \text{ if } f \geqslant 0 \text{ and } \mu(T) = 1\}$$

and we say that its elements are *means* over $T$. The elements of the set

$$V^0 = c_0^V(T)^\perp \cap \mathcal{M}(T)_1^+$$

are called *generalized V-limits* since it is composed of means that extend the limit along $V$.

It is immediately proved that $c_0^V(T)^\perp$ is a lattice vector subspace of $\mathcal{M}(T)$ so that each element $\Psi$ of $c_0^V(T)^\perp$ is of the form

$$\Psi = a\mu_1 - b\mu_2 \quad \text{with } a, b \geqslant 0 \text{ and } \mu_1, \mu_2 \in V^0.$$

From this observation it follows that

$$(*) \qquad f \in c(T) \text{ and } \lim_V f = \alpha \quad \text{iff} \quad \mu(f) = \alpha \text{ for each } \mu \in V^0.$$

Thus the convergence along a filter basis $V$ is a particular case of the following notion of convergence:

Let $\emptyset \neq J \subset \mathcal{M}(T)_1^+$. We say that $f \in C_b(T)$ is *J-convergent* to $\alpha$ if $\mu(f) = \alpha$ for every $\mu \in J$. In such a case we write $J\text{-}\lim f = \alpha$.

One of the main examples is the family $J = [\sigma]$ of Banach limits over $l^\infty$. In this case the $[\sigma]$-convergence is the almost convergence of sequences.

We denote by $J(T)$ (resp., $J(T)_0$) the closed subspace of $C_b(T)$ composed of the $J$-convergent ($J$-convergent to zero) continuous functions.

In the same way we say that $E \in A(T)$ has a *J-density* $\alpha$ if $\mu(E) = \alpha$ for every $\mu \in J$ and we write $d_J(E) = \alpha$.

To each family $J$ the filter basis

$$J^0 = \{E \in Z(T): d_J(E) = 1\}$$

is associated. Given a filter basis $V$, it is easy to prove that

$$V^0 = \{\mu \in \mathcal{M}(T)_1^+ : \mu(M) = 1 \text{ if } M \in V \cap Z(T)\}.$$

In particular, $(J^0)^0 = J^{00} \supset J$.

According to the observation $(*)$, a function $f \in C_b(T)$ is convergent along the filter basis $J^0$ iff it is $J^{00}$-convergent to $\alpha$. Then we say that they are *strongly J-convergent* to $\alpha$. From $J \subset J^{00}$ it follows that if $f \in C_b(T)$ is strongly $J$-convergent to $\alpha$, then $f$ is $J$-convergent to $\alpha$.

We denote by $mJ(T)$ the vector subspace of $J(T)$ composed of its multipliers, that is, those functions $\Phi$ for which $f\Phi \in J(T)$ whenever $f \in J(T)$.

If $\Phi \in mJ(T)$ is such that

$$J\text{-}\lim(\Phi f) = (J\text{-}\lim \Phi)(J\text{-}\lim f) \quad \text{for every } f \in J(T),$$

then we say that $\Phi$ is a *strict multiplier*. We denote by $\bar{m}J(T)$ the subspace of $mJ(T)$ composed of its strict multipliers. They are characterized by the following lemma (by $\tilde{c}$ we denote the constant function $c$ defined on $T$):

LEMMA. *Let $\Phi \in J(T)$. The following are equivalent:*

(1) $\Phi \in \bar{m}J(T)$.

(2) *There exists $c \in \mathbf{R}$ such that* $J\text{-}\lim|\Phi - \tilde{c}| = 0$.

(3) *$\Phi$ is strongly $J$-convergent to $c$.*

Proof. (2)$\Rightarrow$(3). Let $\Psi = \Phi - \tilde{c}$. Let $J\text{-}\lim|\Psi| = 0$ and $\varepsilon > 0$. It is clear that the set

$$Z_\varepsilon = \{t \in T\colon\ |\Psi(T)| \leqslant \varepsilon\}$$

belongs to $J^0 = U$; therefore,

$$\lim_U |\Psi| = 0, \quad \text{i.e.,} \quad \lim_U \Phi = c.$$

(3) ⇒ (2). Let $\Phi$ be strongly convergent to $c$ and $\Psi = \Phi - \tilde{c}$. As $\lim_U \Psi = 0$,

$$\lim_U g\Psi = 0 \quad \text{for every } g \in C_b(T).$$

As $J^{00} \supset J$, $g\Psi$ is $J$-convergent to zero. Let

$$P_1 = \{t \in T\colon\ \Psi(t) > 0\} \quad \text{and} \quad P_2 = \{t \in T\colon\ \Psi(t) < 0\}.$$

Given $\lambda \in J$ and $\varepsilon > 0$, $Z_1 \subset P_1$ and $Z_2 \subset P_2$ exist with $Z_1, Z_2 \in Z(T)$ such that

$$\lambda(P_i \setminus Z_i) < \varepsilon \quad \text{for } i = 1, 2.$$

Then there exists $g \in C_b(T)$ with $|g| \leqslant 1$ such that $g(Z_1) = \{1\}$ and $g(Z_2) = \{-1\}$ (see [10]). For such a function $g \in C_b(T)$ we have $\lambda(\Psi g) = 0$. If $t \in Z_1 \cup Z_2$, then

$$|\Psi(t)| = \Psi(t) g(t),$$

and if we put

$$A = (P_1 \setminus Z_1) \cup (P_2 \setminus Z_2),$$

then

$$\int_T |\Psi(t)|\, d\lambda(t) = \int_T \Psi(t)\, d\lambda(t) - \int_T \Psi(t) g(t)\, d\lambda(t)$$
$$= \int_A \Psi(t)\, d\lambda(t) - \int_A \Psi(t) g(t)\, d\lambda(t).$$

Then

$$\int_T \Psi(t)\, d\lambda(t) < 2\|\Psi\|\, \lambda(A) < 4\varepsilon \|\Psi\|, \quad \text{i.e.,} \quad \lambda(|\Psi|) = 0.$$

Since this is true for every $\lambda \in J$, we obtain (2).

(3) ⇒ (1). If (3) is satisfied and $f \in J(T)$ and if we put

$$\Phi f = (f - c) f + cf,$$

it is clear that (1) is satisfied.

(1) ⇒ (3). If $\Phi \in \bar{m}J(T)$ and $c = J\text{-}\lim \Phi$, let $\Psi = \Phi - \tilde{c}$. Then

$$J\text{-}\lim \Psi^2 = (J\text{-}\lim \Psi)^2 = 0.$$

As we have shown in the proof of the equivalence (2) ⇔ (3), this implies $\lim_U \Psi^2 = 0$, and therefore

$$\lim_U \Phi = c.$$

**3. The main result.** Following the notation of Maddox in [8], the space of real strongly almost convergent sequences is composed of the sequences of $l^\infty$ for which there exists $\alpha \in \boldsymbol{R}$ such that

$$\lim_n \Big( \sum_{i=p+1}^{p+n} |x_i - \alpha| \Big)/n = 0 \quad \text{uniformly in } p \in \boldsymbol{N}.$$

The strongly Cesàro summable sequences are defined as those for which there exists $\alpha \in \boldsymbol{R}$ such that

$$\lim_n \Big( \sum_{i=1}^{n} |x_i - \alpha| \Big)/n = 0.$$

As a consequence of the Lemma and the characterization of Lorentz [7] of the almost convergent sequences, it follows that a sequence $x$ in $l^\infty$ is strongly almost convergent to $\alpha$ iff it is strongly $[\sigma]$-convergent to $\alpha$, that is, it is convergent to $\alpha$ along the basis filter $[\sigma]^0$ composed of the subsets $E \subset N$ for which the uniform density exists and takes the value one:

$$d_{[\sigma]}(E) = \lim_n \Big( \sum_{i=p+1}^{p+n} \chi_E(i) \Big) = 1 \quad \text{uniformly in } p \in N.$$

The Cesàro summable sequences are $[\mathscr{C}]$-convergent according to the family $\mathscr{C} = C^*(\mathscr{F}^0)$, where $\mathscr{F}$ represents the Fréchet filter in $N$, $C$ is the operator

$$C(x)(n) = \big(x(1) + \ldots + x(n)\big)/n,$$

and $C^*$ denotes the transposed operator of $C$.

Then, by the Lemma, the strongly summable Cesàro sequences are precisely the strongly $[\mathscr{C}]$-convergent sequences, that is, those sequences that are convergent along the filter basis $[\mathscr{C}]^0$, composed of the subsets $E \subset N$ for which the usual density $d$ exists and takes the value one:

$$d(E) = \lim_n \Big( \sum_{i=1}^{n} \chi_E(i) \Big)/n = 1.$$

In the following, $T$ will be a paracompact locally compact and non-compact space and $h\colon T \to T$ a proper map, that is, a continuous map for which the inverse image of every compact is a compact. We suppose besides that such a map satisfies the condition that for every compact $K \subset T$ there exists $n = n(K)$ such that $h^n(K) \cap K = \emptyset$. We use $[h]$ to denote the set of means $\mu$ such that $\mu(f \circ h) = \mu(f)$ for every $f \in C_b(T)$. In the following $V$ will be the filter basis $\{T\backslash K: K \text{ compact}\}$.

As $h$ is proper, it is easy to prove that the associated operator $L$, $L(f) = f \circ h$, transforms functions $f$ from $c(T)$ into functions $L(f) = g$ convergent to the same limit as $f$. Since $L^*(V^0) \subset \mathcal{M}(T)_1^+$, the above condition is equivalent to $L^*(V^0) \subset V^0$.

The Markov–Kakutani Theorem implies $[h] \neq \emptyset$.

It is also verified that $[h] \subset V^0$. In fact, it is sufficient to prove that if $\mu \in [h]$, then $\mu(\Phi) = 0$ for every continuous function $\Phi$ with a compact support satisfying $0 \leqslant \Phi \leqslant 1$.

Given the compact $K = \sup \Phi$, there exists $n \in N$ such that $h^n(K) \cap K = \emptyset$. Consequently,

$$0 \leqslant \Phi + \Phi h^n \leqslant 1 \quad \text{and} \quad \mu(\Phi) + \mu(\Phi h^n) = \mu(\Phi) + \mu(\Phi) \leqslant 1,$$

that is, $\mu(\Phi) \leqslant 1/2$. We can apply the same reasoning to the function $\Phi^1 = \Phi + \Phi h^n$, whence we get $\mu(\Phi) \leqslant 1/4$. If we repeat the process, we obtain $\mu(\Phi) = 0$. From $[h] \subset V^0$ it follows that every $h$-invariant mean $\mu \in [h]$ is a generalized $V$-limit, and therefore the $[h]$-convergence is an extension of the convergence along $V$.

Let

$$L_n = \left(\sum_{i=0}^{n-1} L^i\right)/n$$

and let $B$ be the closed vector subspace of $C_b(T)$ generated by the functions $(L-I)\Phi$ with $\Phi \in C_b(T)$.

Proposition. *The following are equivalent:*

(a) $\lim_n \|L_n(f) - \tilde{\alpha}\| = 0$.

(b) $[h]\text{-}\lim f = \alpha$.

(c) $f - \tilde{\alpha}$ *is in* $B$.

Proof. (a) $\Rightarrow$ (b) is evident since $\mu(L_n f) = \mu(f)$ for every $\mu \in [h]$.

(b) $\Rightarrow$ (a). It will be sufficient to prove that $L_n(f) \to \tilde{\alpha}$ weakly and apply the ergodic mean theorem (see [5], VIII–5–1). For this it is sufficient to verify that if $\nu \in \mathcal{M}(T)_1^+$, then

$$\lim_n \nu(L_n f) = \alpha.$$

If $\lambda \in \mathcal{F}^0$, it is easy to verify that $\lambda\text{-}\lim \nu(L_n f) = a(f)$ defines an element $a \in [h]$. Since (b) holds, $\nu(L_n f)$ is a sequence $\mathcal{F}^0$-convergent to $\alpha$, so $\lim_n \nu(L_n f) = \alpha$.

(a) $\Rightarrow$ (c) is a consequence of the application of [5], Corollary VIII–5–2, to this case.

In the following, we will take as the family of means $J$ the family of invariant means $[h]$.

Remark. Let $\delta_t(f) = f(t)$ $(t \in T)$. The sequence of measures $\pi_n = L_n^*(\delta_t)$ is composed of $t$-additive measures since they are discrete:

$$\pi_n(f) = L_n(f)(t) = \big(f(t)+f(h(t))+ \ldots +f(h^{n-1}(t))\big)/n$$
$$= (\delta_t + \delta_{h(t)} + \ldots + \delta_{h^{n-1}(t)})/nf.$$

Applying the Proposition we infer that if $g$ is $[h]$-convergent to $\alpha$, then

$$\lim_n \pi_n(g) = \alpha.$$

THEOREM. *The following are equivalent*:

(a) $\Phi \in mJ(T)$.

(b) $\Phi$ *is strongly $J$-convergent.*

(c) $\Phi \in \bar{m}J$.

Proof. (b)$\Rightarrow$(a) is evident by the Lemma.

(a) $\Rightarrow$ (b). If $\Phi$ is a multiplier of $J(T)$, then the operator $M_\Phi(f) = \Phi f$ will transform $J$-convergent functions into $J$-convergent functions, so that the function $M_\Phi(L-I)f$ will be $J$-convergent.

We can define $\mu \in C_b(T)'$ by

$$\mu(f) = J\text{-lim}\, M_\Phi(L-I)f.$$

By the Remark there exists a sequence $\pi_n$ on $\mathcal{M}_t(T)$ such that the function $g = \Phi(Lf - f)$ satisfies

$$\lim_n \pi_n(g) = \mu(f).$$

Then

$$\mu(f) = \lim_n \gamma_n(f),$$

where

$$\gamma_n = \big(\Phi(t)\,\delta_{h(t)} + \ldots + \Phi(h^{n-1}(t))\,\delta_{h^n(t)}$$
$$- \Phi(t)\,\delta_t - \ldots - \Phi(h^{n-1}(t))\,\delta_{h^{n-1}(t)}\big)/n,$$

and the $\gamma_n$ are also $t$-additive. Applying a result from [9] related to sequences of $t$-additive measures, we infer that $\mu$ is a $t$-additive measure because $T$ is a paracompact and locally compact space, that is, $\mu$ is a Radon measure on $T$. On the other hand, if $f \in c_0^V(T)$, then $\Phi(Lf - f)$ belongs also to $c_0^V(T)$. As $J \subset V^0$, it follows that $\mu(f) = 0$. Since the Radon measure $\mu$ is zero on $c_0^V(T)$, we have $\mu = 0$, so $M_\Phi(Lf - f)$ is $J$-convergent to zero.

Therefore, the operator $M_\Phi$ will transform functions of $B$ into functions $J$-convergent to zero. If $f$ is $J$-convergent to $b$, then, by the Proposition,

$g = f - \tilde{b}$ is in $B$. Consequently,

$$J\text{-lim}\, M_{\Phi} f = J\text{-lim}\, M_{\Phi}(\tilde{h}) = (J\text{-lim}\, f)\,(J\text{-lim}\, \Phi).$$

The application of the Lemma leads us to (a).

(b)$\Leftrightarrow$(c) is obvious by the Lemma.

If $T = N$ and $h = \sigma$, then $[\sigma]$ are the Banach limits and by the application of the Theorem we obtain the results of Ching-Chou [3], Theorem 3.1. In this case the multipliers of the space of the almost convergent sequences are the strongly almost convergent sequences.

Acknowledgment. I am grateful to the referee for many useful suggestions.

## *REFERENCES*

[1] A. Alexandrov, *Additive set-functions in abstract spaces*, Mat. Sb. (N. S.) 8 (50) (1940), pp. 307–348. *Additive set-functions in abstract spaces*, ibidem 9 (51) (1941), pp. 563–628. *Additive set-functions in abstract spaces*, ibidem 13 (55) (1943), pp. 169–238.

[2] F. Balibrea, *Sumabilidad en espacios topológicos. Funciones continuas casi convergentes*, Facultad de Ciencias Químicas y Matemáticas, Universidad de Murcia, 1983.

[3] Ching-Chou, *The multipliers of the space of almost convergent sequences*, Illinois J. Math. 16 (1972), pp. 687–694.

[4] – and J. P. Duran, *Multipliers for the space of almost convergent functions on a semigroup*, Proc. Amer. Math. Soc. 39 (1973), pp. 125–128.

[5] N. Dunford and D. Schwarz, *Linear Operators*, Interscience, New York 1958.

[6] J. P. Duran, *Almost convergence, summability and ergodicity*, Canad. J. Math. 2 (1974), pp. 372–387.

[7] G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. 80 (1948), pp. 167–190.

[8] I. J. Maddox, *A new type of convergence*, Math. Proc. Cambridge Philos. Soc. 83 (1978), pp. 61–64.

[9] W. Moran, *Measures on metacompact spaces*, Proc. London Math. Soc. 20 (1970), pp. 121–142.

[10] V. Varadarajan, *Measures on topological spaces*, Trans. Amer. Math. Soc. (2) 48 (1965), pp. 161–228.

[11] R. Wheeler, *A survey of Baire measures and strict topologies*, Exp. Math. 2 (1983), pp. 97–190.

FACULTAD DE CIENCIAS QUÍMICAS Y MATEMATICAS
UNIVERSIDAD DE MURCIA, SPAIN

*Reçu par la Rédaction le 20. 12. 1985*