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EXTREME CONTRACTIONS ON CERTAIN FUNCTION SPACES

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1. Introduction. Throughout this paper we assume that X and Y are compact Hausdorff topological spaces and Y is *Stonian* (i.e. extremally disconnected). We denote by $C(X)$ and $C(Y)$ the corresponding Banach lattices of real-valued continuous functions. Since $C(Y)$ is order complete, the Banach space $\mathscr{L}(C(X), C(Y))$ of all continuous linear operators from $C(X)$ into $C(Y)$ is a Banach lattice under its canonical ordering (see [7], II.7.7 and IV.1.5).

Let us consider the unit ball U in $\mathscr{L}(C(X), C(Y))$. Elements of U are called *contractions*. A contraction $T \in U$ is said to be *nice* if its adjoint T' takes Dirac measures on Y into extreme points of the unit ball in $C(X)'$. If T is nice, then its adjoint is extreme in the unit ball of $\mathscr{L}(C(Y)', C(X)')$ and all the more so T is extreme in U. In [8] Sharir has shown the reverse implication: any extreme contraction is nice (see also [2] for a shorter proof). A report on related results can be found in [9].

The aim of this note is to present another proof of Sharir's theorem (Section 2), to prove a similar result for order continuous operators (Section 3), and to characterize the extreme contractions on abstract Lebesgue spaces (Section 4). Our approach differs from that of [8] and [2] in that it uses the Banach lattice techniques for $\mathscr{L}(C(X), C(Y))$, avoiding the Stone-Čech compactification. Also, by reducing the problem to positive operators we obtain new equivalent conditions for a contraction to be nice, expressed in terms of its positive and negative parts (Theorem 1). The argument is then modified in Section 3 to obtain a characterization of order continuous contractions on hyperstonian spaces.

After having submitted the first draft of the paper the author learned that a similar idea was recently used by Choo-whan Kim [1] in characterizing the extreme contractions on l^∞ and l^1. Theorems 1 and 2 of [1] can be deduced from our Theorems 1 and 2, and a result of Phelps ([5], Theorem 2.1).

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2. Extreme contractions.

LEMMA 1. *Let $T \in U$ satisfy $0 < T^+1(y) < 1$ and $0 < T^-1(y) < 1$ for some element y in Y. Then there exist two contractions $S_1 \neq S_2$ with $|S_1|, |S_2| \leqslant 2|T|$ and $T = (S_1+S_2)/2$.*

Proof. By the assumption and Urysohn's lemma there exists a continuous function $0 \leqslant w \leqslant 1$ on Y such that $w(y) > 0$ and $(1+w)T^{\pm}1 \leqslant 1$. Letting $u = wT^-1$ and $v = wT^+1$ we get

$$(1 \pm u)T^+ \wedge (1 \mp v)T^- \leqslant 2T^+ \wedge 2T^- = 0.$$

Now put

$$S_1 = (1+u)T^+ - (1-v)T^- \quad \text{and} \quad S_2 = (1-u)T^+ - (1+v)T^-.$$

Clearly, $T = (S_1+S_2)/2$ and $|S_i| \leqslant 2|T|$. The S_i $(i = 1, 2)$ are contractions since

$$\|S_1\| = \||S_1|\| = \||S_1|1\| = \left\|\left((1+u)T^+ + (1-v)T^-\right)1\right\| = \||T|1\| \leqslant 1$$

and, analogously, $\|S_2\| \leqslant 1$. Finally, $S_1 \neq S_2$ since

$$S_1 1(y) - S_2 1(y) = 4u(y)T^+1(y) \neq 0.$$

LEMMA 2. *If $T \in \operatorname{ex} U$, then $|T|1 = 1$.*

Proof. We have $|T|1(y) = 0$ or 1 for all $y \in Y$, since in the contrary case there would exist a function $u \in C(Y)$ with $0 \leqslant u \leqslant 1$ and $u \neq 0$ such that $(1+u)|T|1 \leqslant 1$, implying

$$\|(1 \pm u)T\| = \|(1 \pm u)|T|1\| \leqslant 1$$

and, consequently, $T \notin \operatorname{ex} U$, a contradiction. Now we show that $|T|1 = 1$. Indeed, suppose that the closed and open set $Y_0 = \{y \in Y\colon T1(y) = 0\}$ is non-empty and denote its characteristic function by χ. For any functional μ on $C(X)$ with $0 < \|\mu\| \leqslant 1$ we have a non-zero operator $\mu \otimes \chi$: $C(X) \to C(Y)$ defined by $(\mu \otimes \chi)f(y) = \langle \mu, f \rangle \chi(y)$. Since $|Tf| \leqslant |T|f \leqslant \|f\| |T|1$ for any positive f in $C(X)$, we have $Tf(y) = 0$ whenever $y \in Y_0$. Therefore, $\|T \pm \mu \otimes \chi\| \leqslant 1$, contradicting $T \in \operatorname{ex} U$.

Let P be the convex set of all positive contractions in $\mathscr{L}\left(C(X), C(Y)\right)$, and P_1 its subset consisting of all positive contractions which take 1 into 1. Let us recall the well-known characterization of $\operatorname{ex} P_1$, valid for arbitrary compact Hausdorff spaces X and Y (see [3], [5], Theorem 2.1, and [7], III.9.1 and 9.2):

(o) $T \in \operatorname{ex} P_1$ *if and only if there exists a continuous map φ: $Y \to X$ such that $Tf(y) = f\left(\varphi(y)\right)$ for all $f \in C(X)$ and all $y \in Y$.*

The essential part (i) $\Leftrightarrow$ (v) of the forthcoming theorem is due to Sharir ([8], Theorem 2).

THEOREM 1. *For any* $T \in \mathscr{L}(C(X), C(Y))$ *the following conditions are equivalent*:

(i) $T \in \operatorname{ex} U$;

(ii) $T^{\pm} \in \operatorname{ex} P$ *and* $|T1| = 1$;

(iii) $|T| \in \operatorname{ex} P_1$;

(iv) *there exist a function* $r \in C(Y)$ *with* $|r| = 1$ *and a continuous map* $\varphi\colon Y \to X$ *such that* $Tf(y) = r(y)f(\varphi(y))$ *for all* $f \in C(X)$ *and all* $y \in Y$;

(v) T *is nice.*

Proof. (i) $\Rightarrow$ (ii). By Lemmas 1 and 2, $T^{\pm}1$ are characteristic functions and $\{\operatorname{supp} T^{+}1, \operatorname{supp} T^{-}1\}$ is a partition of Y into two closed and open subsets. Suppose, say, that $T^{+} = (S_1 + S_2)/2$ for some $S_i \in P$. We have $S_i \leqslant 2T^{+}$, so that

$$S_i \wedge T^{-} = 0 \quad \text{and} \quad \|S_i - T^{-}\| = \|S_i + T^{-}\| = \|S_i 1 + T^{-}1\| \leqslant 1.$$

Since $T = \big((S_1 - T^{-}) + (S_2 - T^{-})\big)/2$, we must have $S_1 = S_2$. Analogously we show $T^{-} \in \operatorname{ex} P$.

(ii) $\Rightarrow$ (iii). Arguing as in the proof of Lemma 2 we see that any operator from $\operatorname{ex} P$ maps 1 into a characteristic function in $C(Y)$. Therefore, $T^{\pm}1$ are characteristic functions and we have

$$1 \geqslant |T|1 = T^{+}1 + T^{-}1 \geqslant |T1| = 1.$$

It follows that if $S_i \in P$ and $|T| = (S_1 + S_2)/2$, then

$$T^{\pm} = \big((T^{\pm}1)S_1 + (T^{\pm}1)S_2\big)/2,$$

so that $(T^{\pm}1)S_i = T^{\pm}$. Hence $S_i = |T|$.

(iii) $\Rightarrow$ (iv). For any $S \in \mathscr{L}(C(X), C(Y))$ and $y \in Y$ we define a continuous linear functional $S'(y)$ on $C(X)$ by the formula $S'(y)f = Sf(y)$. Clearly, $S'(y) = S'\delta_y$, where δ_y denotes the Dirac measure concentrated at y.

By (o) there exists a continuous map $\varphi\colon Y \to X$ such that $|T|f(y) = f(\varphi(y))$ for all $f \in C(X)$ and all $y \in Y$. In particular, $|T|'(y)$ is the Dirac measure $\delta_{\varphi(y)}$ for each $y \in Y$. Denoting by $|T'(y)|$ the modulus of $T'(y)$ in the Banach lattice $C(X)'$, we have $|T'(y)| \leqslant |T|'(y)$, so that there exists a real number $r(y)$ with $|r(y)| \leqslant 1$ and $T'(y) = r(y)\,\delta_{\varphi(y)}$. Therefore, $Tf(y) = r(y)f(\varphi(y))$ for all $f \in C(X)$. Letting $f = 1$, we get $r = T1 \in C(Y)$. Since $|Tf| \leqslant |r|\,|T|f$ for all $f \in C(X)$, $f \geqslant 0$, we have $|r| = 1$.

(iv) $\Rightarrow$ (v) $\Rightarrow$ (i) are obvious.

By the Stone-Weierstrass theorem, for each $x \in X$ the functions $f \in C(X)$ satisfying $\|f\| \leqslant 1$ and $|f(x)| = 1$ form a linearly dense subset of $C(X)$. Thus any operator of the form (iv) necessarily satisfies

(vi) *for each* $y \in Y$ *the set* $\{f \in C(X)\colon \|f\| \leqslant 1$ *and* $Tf(y) = 1\}$ *is linearly dense in* $C(X)$.

This apparently weak condition is, in fact, sufficient for a contraction T to be of the form (iv), as follows from Theorem 2 of [6], where this result is proved in a more general setting.

3. Order continuous contractions. Throughout this section we assume that X and Y are ***hyperstonian spaces***, which means that the Banach lattices $\mathcal{N}(X)$ and $\mathcal{N}(Y)$ of all order continuous Radon measures on X and Y, respectively, separate $C(X)$ and $C(Y)$. The latter spaces can now be viewed as dual Banach lattices of $\mathcal{N}(X)$ and $\mathcal{N}(Y)$ (see [7], II.9.3). The space $\mathcal{L}(\mathcal{N}(Y), \mathcal{N}(X))$ is a Banach lattice under its canonical ordering ([7], IV.1.5 (ii), III.11.4, and II.8.5).

Let us recall that a net (f_a) in $C(X)$ *order converges* to f if and only if there exists a downward directed family (g_a) in $C(X)$ such that

$$|f_a - f| \leqslant g_a \quad \text{and} \quad \inf_a g_a = 0$$

(cf. [7], II.1.7 and II, Exercise 2 (a)). An operator $T\colon C(X) \to C(Y)$ is said to be *order continuous* if (Tf_a) order converges to Tf whenever (f_a) order converges to f (see [7], II.2.4). We denote by $\mathcal{L}_0$ the set of all order continuous operators in $\mathcal{L}(C(X), C(Y))$.

LEMMA 3. *Let X and Y be hyperstonian spaces. Then the set $\mathcal{L}_0$ forms in the space $\mathcal{L}(C(X), C(Y))$ an ideal which is Banach lattice isomorphic to $\mathcal{L}(\mathcal{N}(Y), \mathcal{N}(X))$ under the mapping $S \to S'$.*

Proof. First we show that S' is in $\mathcal{L}_0$ whenever S is a positive operator in $\mathcal{L}(\mathcal{N}(Y), \mathcal{N}(X))$. Indeed, let (f_a) and (g_a) be such as in the definition of order convergence. The family $(S'g_a)$ is also downward directed and $C(Y)$ is order complete, so there exists $h = \inf S'g_a$ in $C(Y)$. Now, for any positive functional ν in $\mathcal{N}(Y)$ we have

$$\langle \nu, h\rangle = \inf \langle \nu, S'g_a\rangle = \inf\langle S\nu, g_a\rangle = 0,$$

whence $h = 0$ and S' is order continuous.

Since the sum of two order continuous operators is itself order continuous, $\mathcal{L}_0$ is a vector subspace of $\mathcal{L}(C(X), C(Y))$. Since, moreover, $\mathcal{L}(\mathcal{N}(Y), \mathcal{N}(X))$ is a vector lattice, we infer that all operators of the form S' are in $\mathcal{L}_0$. Conversely, if $T \in \mathcal{L}_0$, then T' takes $\mathcal{N}(Y)$ into $\mathcal{N}(X)$, so that T is the adjoint of the restriction S of T' to $\mathcal{N}(Y)$. So far we have shown that $T \in \mathcal{L}(C(X), C(Y))$ is order continuous if and only if it has a pre-adjoint.

Clearly, the map $S \to S'$ is a linear isometry. In order to prove that it is a Banach lattice isomorphism it suffices to show that $|S'| = |S|'$ for any $S \in \mathcal{L}(\mathcal{N}(Y), \mathcal{N}(X))$. The obvious inequality $|S| \geqslant \pm S$ implies $|S|' \geqslant \pm S'$, whence $|S|' \geqslant |S'|$. By the last inequality, $|S'|$ is order continuous along with $|S|'$, whence, by the first part of the proof, $|S'| = S_0'$ for some positive operator S_0. This implies $\pm S' \leqslant S_0'$. By taking the second adjoints and restricting them to $\mathcal{N}(Y)$, we obtain $\pm S \leqslant S_0$, so that $|S| \leqslant S_0$ and $|S|' \leqslant S_0' = |S'|$.

To conclude the proof we note that if a positive operator T_0 is in $\mathcal{L}_0$ and an operator T in $\mathcal{L}(C(X), C(Y))$ satisfies $|T| \leqslant T_0$, then $T \in \mathcal{L}_0$. Hence $\mathcal{L}_0$ is an ideal in $\mathcal{L}(C(X), C(Y))$.

Let X and Y be hyperstonian spaces. We denote by U_0 the set of all order continuous contractions in $\mathcal{L}(C(X), C(Y))$.

PROPOSITION. $\operatorname{ex} U_0 = U_0 \cap \operatorname{ex} U$.

Proof. We only need to show that $\operatorname{ex} U_0 \subset \operatorname{ex} U$. Taking μ in the proof of Lemma 2 to be order continuous, we can see that the lemma still holds under the assumption $T \in \operatorname{ex} U_0$. Also, in the proof of (i) $\Rightarrow$ (ii) of Theorem 1, the contractions S_i satisfy $|S_i| \leqslant 2|T|$, so that they are order continuous if T is (by Lemma 3). Hence we obtain (ii) of Theorem 1 provided that $T \in \operatorname{ex} U_0$. Therefore, $T \in \operatorname{ex} U_0$ implies $T \in \operatorname{ex} U$.

The following corollary is a consequence of Theorem 1 and the Proposition.

COROLLARY. *Let X, Y be hyperstonian spaces and let $T \in \mathcal{L}(C(X), C(Y))$. Then $T \in \operatorname{ex} U_0$ if and only if there exist a function $r \in C(Y)$ with $|r| = 1$ and a continuous open map φ: $Y \to X$ such that $Tf(y) = r(y) f(\varphi(y))$ for all $f \in C(X)$ and all $y \in Y$.*

Proof. By [7], III.9.3 (where u and v are the constant one functions), the operator of the form $f \to f \circ \varphi$ is order continuous if and only if φ is an open map. Hence the assertion follows from the equivalence (i) $\Leftrightarrow$ (iv) of Theorem 1 and from the Proposition.

4. Contractions on AL-spaces. A Banach lattice E is called an *AL-space* (*abstract Lebesgue space*) if the norm in E is additive on the positive cone. By the Kakutani representation theorem, each AL-space is isomorphic to $L^1(\mu)$ for some positive Radon (not necessarily bounded) measure μ on a locally compact space. Another representation theorem asserts that E can be identified with the lattice $\mathcal{N}(Y)$ of all order continuous Radon measures on a (unique up to homeomorphism) compact hyperstonian space Y (see [7], II.8.5 and 9.2).

Let now E and F be AL-spaces, and Y and X the associated hyperstonian spaces. Denoting by V the unit ball of $\mathcal{L}(E, F)$ we have, in view of Lemma 3, $V' = U_0$ and, as a consequence of the Proposition, $(\operatorname{ex} V)' = \operatorname{ex} U_0 = V' \cap \operatorname{ex} U$. Thus, the Corollary gives a characterization of extreme contractions on AL-spaces in terms of their adjoints acting between the corresponding spaces of continuous functions. In measure theory, however, where AL-spaces usually occur as concrete L^1-spaces for σ-finite measures, it seems more natural to identify the duals of E and F with the corresponding L^∞-spaces rather than the spaces of continuous functions. Unfortunately, in that case the representation of an extreme contraction $T \in \mathscr{L}(E, F)$ by means of a measurable transformation φ is not always possible, as is shown by the following example, communicated to the author by Professor C. Ryll-Nardzewski.

Example. Let λ be the Haar measure on the (multiplicative) circle group Γ in the complex plane. There exists a non-measurable subset Q_0 of Γ such that

$$Q_0 \cap Q_1 = \emptyset, \quad Q_0 \cup Q_1 = \Gamma, \quad \text{and} \quad \lambda^*(Q_0) = \lambda^*(Q_1) = 1,$$

where $Q_1 = \{z \in \Gamma\colon -z \in Q_0\}$. Let $E = L^1(\lambda)$ and $F_i = L^1(\mu_i)$, where μ_i is the restriction of λ to Q_i ($i = 0, 1$). We denote by I_i the canonical isomorphism from $E' = L^\infty(\lambda)$ onto $L^\infty(\mu_i)$ defined by $I_i f = f|Q_i$. Let S be the Banach lattice automorphism of E' defined by $Sf(x) = f(-x)$. The composition $I_i S I_i^{-1}$, being an automorphism of F_i', is an extreme contraction (by the Banach-Stone theorem and (iv) $\Rightarrow$ (i) of Theorem 1). Since it is also order continuous, by Lemma 3 we have $I_i S I_i^{-1} = T_i'$ for some extreme contraction T_i in $\mathscr{L}(F_i, F_i)$. Now we show that T_0 is not induced by any measurable transformation of Q_0. Indeed, if T_0 were induced by φ_0, then, by the symmetry, T_1 would be induced by a transformation φ_1 of Q_1. Then $x \to (\varphi_0 \cup \varphi_1)(x)$ and $x \to -x$ would induce the same automorphism S of E', which is a contradiction, since $\varphi_i(x) \neq -x$ for all $x \in \Gamma$.

Nevertheless, the following result is valid:

THEOREM 2. *Let (Q, Σ, μ) be a σ-finite measure space and let ν be a σ-finite Borel measure on the real line R. Then $T \in \mathscr{L}\big(L^1(\mu), L^1(\nu)\big)$ is an extreme contraction if and only if there exist $r \in L^\infty(\mu)$ with $|r| = 1$ and a non-singular measurable transformation $\varphi\colon Q \to R$ such that the adjoint T' of T is of the form*

$$T'f(y) = r(y) f\big(\varphi(y)\big) \ \text{a.e.} \quad \text{for all } f \in L^\infty(\nu).$$

(Here *non-singular* means $\nu(B) = 0 \Rightarrow \mu\big(\varphi^{-1}(B)\big) = 0$.)

Proof. Let X and Y be the associated hyperstonian spaces of $F = L^1(\nu)$ and $E = L^1(\mu)$, respectively.

Sufficiency. The non-singular transformation $\varphi\colon Q \to R$ induces an order continuous Banach lattice homomorphism $T_\varphi f = f\circ\varphi$ from $L^\infty(\nu)$ into $L^\infty(\mu)$. The Banach lattices $L^\infty(\nu)$ and $C(X)$, as well as $L^\infty(\mu)$ and $C(Y)$, are canonically isomorphic (as dual Banach lattices of F and E, respectively), hence to T_φ there corresponds an order continuous Banach lattice homomorphism $T_0 \in \mathscr{L}\big(C(X), C(Y)\big)$. In view of [7], III.9.1 and 9.3, T_0 is induced by a continuous and open map $\varphi_0\colon Y \to X$. Also, to r there corresponds a function r_0 in $C(Y)$ with $|r_0| = 1$. By the Corollary, the operator $T_1 f = r_0(f\circ\varphi_0)$ is an extreme order continuous contraction in $\mathscr{L}(C(X), C(Y))$. We can identify T_1 with T' so, by Lemma 3, T is an extreme contraction in $\mathscr{L}(E, F)$.

Necessity. By Lemma 3, T' is order continuous. Hence, in view of the Corollary, $T' = rP$ for some order continuous $P \in \mathscr{L}(L^\infty(\nu), L^\infty(\mu))$ with $|Pf| = P|f|$ and $P1 = 1$. Now we apply Lemma 3 in [4].

REFERENCES

[1] Choo-whan Kim, *Extreme contraction operators on l^∞*, Mathematische Zeitschrift 151 (1976), p. 101-110.

[2] A. Gendler, *Extreme operators in the unit ball of $L(C(X), C(Y))$ over the complex field*, Proceedings of the American Mathematical Society 57 (1976), p. 85-88.

[3] A. Ionescu Tulcea and C. Ionescu Tulcea, *On the lifting property I*, Journal of Mathematical Analysis and Applications 3 (1961), p. 537-546.

[4] A. Iwanik, *Pointwise induced operators on L_p-spaces*, Proceedings of the American Mathematical Society 58 (1976), p. 173-178.

[5] R. R. Phelps, *Extreme positive operators and homomorphisms*, Transactions of the American Mathematical Society 108 (1963), p. 265-274.

[6] R. Sato, *Operators on continuous function spaces and L_1-spaces*, Colloquium Mathematicum 34 (1976), p. 257-269.

[7] H. H. Schaefer, *Banach lattices and positive operators*, Berlin 1974.

[8] M. Sharir, *Characterizations and properties of extreme operators into $C(Y)$*, Israel Journal of Mathematics 12 (1972), p. 174-183.

[9] — *A counterexample on extreme operators*, ibidem 24 (1976), p. 320-328.

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