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ON THE DEFICIENCY OF PRODUCT SPACES

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All spaces are assumed to be separable metrizable. As usual, the *deficiency of a space Z*, $\operatorname{def} Z$, is defined to be the integer

$$\min\{\dim[K-Z] \mid K \text{ is a compactification of } Z\}.$$

Recently, Aarts proved the following interesting theorem (see [1], Theorem 3) giving lower bounds for the deficiency of product spaces:

THEOREM. *If Y is lacunary (i.e., nowhere locally compact, or, equivalently, $\overline{Y-C} = Y$ for any compact subset C of Y) and if Y is of the second category, then*

$$\operatorname{def}(X \times Y) \geqslant \dim X \quad \textit{for any space } X.$$

On the other hand, Lelek proved a theorem (see [2], Theorem 2) which can be formulated as follows:

THEOREM. *If f is a continuous mapping of a space X onto a G_δ (i.e., topologically complete) lacunary space Y such that the set $f^{-1}(y)$ is compact for each y in Y, then*

$$\operatorname{def} X \geqslant \min\{\dim f^{-1}(y) \mid y \in Y\}.$$

These two results led Lelek to pose the following question:

QUESTION. If f is an (open) continuous mapping of a space X onto a lacunary G_δ space Y, is it true that

$$\operatorname{def} X \geqslant \min\{\dim f^{-1}(y) \mid y \in Y\}?$$

In this paper, we answer this question in the affirmative and, therefore, generalize both of the above-mentioned theorems.

THEOREM. *If f is a continuous mapping of a space X onto a space Y which cannot be represented as a countable union of its compact subsets, then*

$$\operatorname{def} X \geqslant \min\{\dim f^{-1}(y) \mid y \in Y\}.$$

Proof. Let Z be a compactification of X such that $\dim[Z-X]$

$= \operatorname{def} X$. By Theorem II.10 of [3], there exists a G_δ-set G of Z such that $Z - X \subset G$ and $\dim G = \dim[Z - X]$. Hence $Z - G$ is F_σ in Z, so that

$$Z - G = \bigcup_{i=1}^{\infty} F_i,$$

where each F_i is a compact subset of Z. By the assumption on Y, there is a $y \in Y - \bigcup_{i=1}^{\infty} f(F_i)$ so that

$$f^{-1}(y) \subset f^{-1}\Big[Y - \bigcup_{i=1}^{\infty} f(F_i)\Big] \subset X - \bigcup_{i=1}^{\infty} F_i \subset Z - \bigcup_{i=1}^{\infty} F_i = G,$$

and, therefore,

$$\dim f^{-1}(y) \leqslant \dim G.$$

Thus

$$\operatorname{def} X \geqslant \min\{\dim f^{-1}(y) \mid y \in Y\}.$$

COROLLARY. *If f is a continuous mapping of a space X onto a lacunary space Y of the second category, then*

$$\operatorname{def} X \geqslant \min\{\dim f^{-1}(y) \mid y \in Y\}.$$

Proof. The Corollary follows from the Theorem, since, in this case, Y cannot be a countable union of its compact subsets. For supposing that

$$Y = \bigcup_{i=1}^{\infty} F_i,$$

where each F_i is a compact subset of Y, then, since Y is lacunary, each F_i is nowhere dense in Y so that Y is actually of the first category which contradicts the assumption on Y.

Remark. Since topologically complete spaces are of the second category, Lelek's Theorem follows immediately from the Corollary.

Finally, Aarts' Theorem also follows easily from the Corollary if we utilize the natural projection from the product space.

REFERENCES

[1] J. M. Aarts, *On the dimension of remainders in extensions of product spaces*, Colloquium Mathematicum 22 (1971), p. 233-238.

[2] A. Lelek, *Sur deux genres d'espaces complets*, ibidem 8 (1961), p. 31-34.

[3] J. Nagata, *Modern dimension theory*, Groningen 1965.

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