COLLOQUIUM MATHEMATICUM

VOL. LII 1987 FASC. 2

ON THE STATIONARY SOLUTIONS OF BURGERS' EQUATION

BY

PIOTR BILER (WROCŁAW)

Introduction. The system of two equations

$$\dot{U}(t) = p - \nu U(t) - \int_0^\pi v^2(x, t)\, dx, \tag{1}$$

$$v_t(x, t) = \nu v_{xx}(x, t) - \big(v^2(x, t)\big)_x + U(t) v(x, t), \tag{2}$$

where $p, \nu \geqslant 0$, $x \in [0, \pi]$, $t > 0$, $U\colon [0, \infty) \to \boldsymbol{R}$, $v\colon [0, \pi] \times [0, \infty) \to \boldsymbol{R}$, with boundary and initial conditions

$$\begin{gathered} U(0) = U_0, \\ v(x, 0) = \varphi(x) \text{ for } x \in [0, \pi], \quad v(0, t) = v(\pi, t) = 0 \text{ for } t > 0 \end{gathered} \tag{3}$$

was introduced by Burgers in [1] and [2] as a simplified model of fluid motion in order to explain some properties of turbulent flow.

One can interpret U as a velocity of "primary" motion of a viscous fluid in a sloped channel, v as a velocity of "secondary" (turbulent) motion in the direction perpendicular to the axis of the channel, and p as average constant pressure. The physical meaning of the terms in (1), (2) can be found in [2].

Dłotko proved recently ([4]–[6]) the existence, uniqueness and regularity of global solutions of the system (1)–(3) in several different regularity classes. In particular, he proved that the regular solution $v \in C^{2,1}\big((0, \pi) \times (0, \infty)\big)$, continuous up to the boundary, exists for $\varphi \in C_0^1[0, \pi]$ (the proof is based on the theory of solutions of non-linear parabolic equations in Hölder classes). He also showed, using energy methods, that the "laminar" solution $(U, v) = (p/\nu, 0)$ is globally stable for $p/\nu^2 \leqslant 1$.

We can eliminate one of the two parameters p, ν substituting U/ν, v/ν, νt for U, v, t, respectively, if only $\nu \neq 0$. Writing, as before, U, v, t, we obtain the following slightly simpler system:

$$\dot{U} = c - U - z, \tag{4}$$

$$v_t = v_{xx} - (v^2)_x + Uv, \tag{5}$$

where

$$c = p/v^2, \quad z(t) = \int_0^\pi v^2(x, t)\,dx.$$

If $v = 0$, then (in the old variables)

$$\dot{U} = p - z, \tag{6}$$

$$v_t = -(v^2)_x + Uv. \tag{7}$$

Stationary solutions. In the case of stationary solutions of (4), (5): $U = \text{const}$, $v = v(x)$, the equations under consideration are

$$c = U + z, \tag{8}$$

$$v_{xx} - (v^2)_x + Uv = 0 \tag{9}$$

with the conditions

$$v(0) = v(\pi) = 0. \tag{10}$$

It suffices to consider only (9), (10) with $U = \lambda$ as a parameter. We write (9) in the form

$$v'' - (v^2)' + \lambda v = 0. \tag{11}$$

We can try to solve this equation by means of quadratures, e.g., by using the Hopf–Cole substitution $v = -(\log\psi)'$ (see [7] and [3]). We get

$$x = \int_{\psi(0)}^{\psi(x)} [(a - \lambda\log\psi)\psi^2 - a]^{-1/2}\,d\psi,$$

where a is an appropriate constant, chosen according to (10). However, it is not easy to say anything about the solutions if they are written in the above form. Note that Burgers has integrated (11) in another way but his description of the properties of solutions is neither precise nor complete ([1], [2], see also [8]).

We start out to investigate (11) studying the vector field associated with (11): $v' = w$, $w' = v(2w - \lambda)$. This vector field has an (analytic) integral of the form

$$F(v, w, \lambda) = v^2 - [w + (\lambda/2)\log(\lambda - 2w)]. \tag{12}$$

The integral curves are described by the equation $F(v, w, \lambda) = \text{const}$, λ fixed. We can find a solution of (10), (11) only in the region $w < \lambda/2$ because for $w \geqslant \lambda/2$ any integral curve starting from $v = 0$ will never reach $v = 0$ once more. It is easy to see that $\{(v, w)\colon F(v, w, \lambda) = \text{const},\ w < \lambda/2,\ \lambda \text{ fixed}\}$ is either the point (0, 0) or a closed curve symmetric with respect to the w-axis, since F is a strictly increasing function along an arbitrary ray starting from (0, 0). Thus the solution must consist of some parts, half or the whole, eventually taken several times, of an orbit of the vector field, parametrized so that $v(0) = v(\pi) = 0$. In other words: if T is the period of this orbit, then we can construct a solution of (10), (11) (with $2\pi/T-1$ zeros in $(0, \pi)$) if and only if $\pi = (m-1/2)\,T$ or $\pi = mT$ for an integer $m \geqslant 1$. For any such solution v we have the corresponding symmetric solution $\tilde{v}$ which can be written in the form

$$\tilde{v}(x) = -v(\pi - x),$$

$$\tilde{v}(x) = \begin{cases} v\big(x+\pi(2m)^{-1}\big) & \text{for } x \leqslant \pi\big(1-(2m)^{-1}\big), \\ v\big(x-\pi\big(1-(2m)^{-1}\big)\big) & \text{for } x > \pi\big(1-(2m)^{-1}\big), \end{cases}$$

respectively. Thus, the stationary solutions of Burgers' equation always appear in pairs.

One of the ways of solving our problem is to apply the bifurcation theory. Equation (11) with condition (10) can be converted, by using the Green function, into the operator equation in a Banach space:

$$(I-\lambda G)\,v + F(v) = 0. \tag{13}$$

The compact operators G and F in $C_0^1[0, \pi]$ are defined as follows:

$$Gv(x) = \int_0^\pi G(x, y)\,v(y)\,dy,$$

$$F(v)(x) = \int_0^\pi G(x, y)\big(v^2(y)\big)'\,dy,$$

where $G(x, y)$ determined as

$$G(x, y) = \begin{cases} (\pi-y)\,x/\pi & \text{for } 0 \leqslant x \leqslant y \leqslant \pi, \\ (\pi-x)\,y/\pi & \text{for } 0 \leqslant y \leqslant x \leqslant \pi \end{cases}$$

is the Green function for the operator $-d^2/dx^2$ considered in $[0, \pi]$ with homogeneous boundary conditions. G is linear, F — non-linear, $F(0) = 0$; moreover, F is differentiable and its Fréchet derivative is

$$DF(v)(h)(x) = 2\int_0^\pi G(x, y)\big(v(y)\,h(y)\big)'\,dy,$$

so $DF(0) = 0$. The only points (λ, v) in $\boldsymbol{R}\times C_0^1[0, \pi]$ from which non-trivial solutions of (13) can bifurcate are the points $(\lambda_n, 0)$, where $\lambda_n = n^2$ (n — integer, $n \geqslant 1$) is the (simple) characteristic value of the operator G, i.e., the characteristic value of equation (13) linearized at $v = 0$ (see comments before Theorem 3.3.1 in [9]). Theorem 7.5.2 in [10] (Krasnosel'skiĭ's theorem in a new setting due to Crandall and Rabinowitz) assures that $(\lambda_n, 0)$ are actually the bifurcation points: (13) has non-trivial solutions in a suitably small rectangle $\{(\lambda, v)\colon |\lambda-\lambda_n| < \delta,\ \|v\| < \varepsilon\}$. This is the local result. The global theorem due to Rabinowitz (see [10]) states: if λ_n is a simple characteristic value of G, then there is a maximal closed, connected set of points (λ, v) satisfying (13) which tends to infinity in $\boldsymbol{R}\times C_0^1[0, \pi]$. This set consists of two distinct branches of solutions of (13) which meet only at $(\lambda_n, 0)$.

Now we have the complete description of the stationary solutions of Burgers' equation. Pairs of branches consisting of solutions $\{(\lambda, v_\lambda)\colon \lambda > n^2\}$, $\{(\lambda, \tilde{v}_\lambda)\colon \lambda > n^2\}$ bifurcate from the points $(n^2, 0)$ and tend to infinity without intersections with the λ-axis, except $(n^2, 0)$ or other branches. The non-trivial solutions lying on the first bifurcating branch are stable (in the linearized sense) near $\lambda_1 = 1$. The zero solution loses its stability when λ increases and crosses 1. Those results concerning stability of bifurcated solutions follow from a theorem of Crandall and Rabinowitz (3.6 in [9]).

Monotonicity properties. We formulate now some additional properties of the solutions of (11). Due to symmetry we may restrict ourselves to those solutions for which $v'(0) > 0$.

THEOREM. *For $n^2 < \lambda < \mu$ the solutions v_λ, v_μ of (10), (11), which bifurcate from $(n^2, 0)$ and remain on the n-th branch satisfy the following monotonicity condition: $|v_\lambda(x)| \leqslant |v_\mu(x)|$ for any x.*

Clearly, it suffices to show that $v_\lambda(x) < v_\mu(x)$ for $x\in(0, \pi/n)$.

We have immediately the following

COROLLARY. *For $\lambda < \mu$,*

$$z_\lambda = \int_0^\pi v_\lambda^2 < z_\mu = \int_0^\pi v_\mu^2,$$

and therefore the number of the stationary solutions of (4), (5) for a given c, $c = \lambda + z_\lambda$, is equal to $2c^{1/2}-1$ for any integer $c^{1/2}$ and to $2[c^{1/2}]+1$ otherwise.

The proof of the Theorem is based on the following remarks:

(i) If $\lambda < \mu$ and v_λ, v_μ are the solutions of (11) with $\max v_\lambda = \max v_\mu$, then for x, y such that $v_\lambda(x) = v_\mu(y) > 0$ and $\operatorname{sgn} v'_\lambda(x) = \operatorname{sgn} v'_\mu(y)$ the condition $|v'_\lambda(x)| \leqslant |v'_\mu(y)|$ is satisfied and the equality is possible only for $v' = 0$.

This result is more visible on the phase plane: the orbit corresponding to μ contains in its interior the λ-orbit (the same is true for $\max v_\lambda \leqslant \max v_\mu$).

(ii) If $T_+(A, \lambda)$ and $T_-(A, \lambda)$ are the least positive x's such that $v_\lambda(T_+(A, \lambda)) = \max v_\lambda = A^{1/2}$, $v_\lambda(0) = 0$, $v_\lambda(T_+(A, \lambda) + T_-(A, \lambda)) = 0$ (one can interpret $T_\pm$ as time necessary for v to reach $A^{1/2}$, and then to return to 0), then

(a) $T_\pm$ is a strictly decreasing function of λ for A fixed,

(b) $(T_+ + T_-)$ is a strictly increasing function of A for λ fixed.

P r o o f o f t h e T h e o r e m. It follows from (ii) that if $\lambda < \mu$ and v_λ, v_μ are the solutions of the boundary value problem (10), (11), then $\max v_\lambda$

$< \max v_\mu$. If $v_\lambda(x_0) = v_\mu(x_0)$ for $x_0 \in (0, \pi/n)$ and x_0 is either the least x with this property before v_μ reaches its maximum or the greatest x with this property after v_μ reaches its maximum, then $v'_\lambda(x_0) \geqslant v'_\mu(x_0) > 0$ or $v'_\lambda(x_0) \leqslant v'_\mu(x_0) < 0$, respectively. These additional conditions may be satisfied because $v'_\mu(0) > v'_\lambda(0) > 0$ and $v'_\mu(\pi/n) < v'_\lambda(\pi/n) < 0$ as we see from (i). However, the inequalities obtained contradict the property of the phase portrait described in (i): the equality $v_\lambda(x_0) = v_\mu(x_0)$ would imply $|v'_\lambda(x_0)| < |v'_\mu(x_0)|$.

Now we prove the auxiliary facts (i) and (ii).

(i) If the orbits are described by the equations $F(v_\lambda, w_\lambda, \lambda) = a_\lambda$ and $F(v_\mu, w_\mu, \mu) = a_\mu$ (F is the integral (12)) and $\max v_\lambda = \max v_\mu$, then

$$a_\lambda + (\lambda/2)\log\lambda = a_\mu + (\mu/2)\log\mu$$

and

$$0 > v_\lambda^2 - v_\mu^2 = (\lambda/2)\log(1-2w/\lambda) - (\mu/2)\log(1-2w/\mu),$$

since the function $\lambda\log(1-2w/\lambda)$ of λ is strictly increasing for w ($0 \neq w < \lambda/2$) fixed.

(ii) Evidently,

$$T = \int_0^T 1 = \int_0^{v(T)} \frac{dv}{v'},$$

so we express v' as a function of v, A, and λ. Let $F(v, w, \lambda) = a$ be the equation describing the orbit with $\max v = A^{1/2}$, i.e.,

$$v^2 - a = w + (\lambda/2)\log(\lambda - 2w), \qquad A = a + (\lambda/2)\log\lambda.$$

If we put $w = \lambda t$, then $(v^2 - A)/\lambda = h(t)$, where $h(t) = t + 2^{-1}\log(1-2t)$. It is obvious that for every $y < 0$ there exist a unique $s = s(y) < 0$ and a unique $t = t(y) > 0$ such that $h(s) = h(t) = y$; moreover, $t < 1/2$. Using the functions t and s we can write

$$\begin{aligned} T_+(A, \lambda) &= \int_0^{A^{1/2}} \left[\lambda t\left((v^2 - A)/\lambda\right)\right]^{-1} dv, \\ T_-(A, \lambda) &= -\int_0^{A^{1/2}} \left[\lambda s\left((v^2 - A)/\lambda\right)\right]^{-1} dv \end{aligned} \tag{14}$$

or, changing the variables,

$$T_+(A, \lambda) = \int_{-1}^{0} A^{1/2}[2\lambda t(Ar/\lambda)]^{-1}(r+1)^{-1/2}\,dr,$$

$$T_-(A, \lambda) = -\int_{-1}^{0} A^{1/2}[2\lambda s(Ar/\lambda)]^{-1}(r+1)^{-1/2}\,dr.$$

In order to prove (a) it is sufficient to show that $\lambda t(Ar/\lambda)$ and $-\lambda s(Ar/\lambda)$ are increasing functions of λ. We calculate

$$\frac{d(\lambda t(Ar/\lambda))}{d\lambda} = t(Ar/\lambda)-(Ar/\lambda)t'(Ar/\lambda)$$

$$= t-h(t)(1-(2t)^{-1}) = \left[1+((2t)^{-1}-1)\log(1-2t)\right]/2$$

($y = Ar/\lambda = h(t)$, $t'(y) = 1-(2t)^{-1}$), which is positive for $t > 0$ and negative for $t < 0$ (apply the inequality $\log(1+x) < x$ with $x = -1+(1-2t)^{-1}$).

Case (b) is more complicated since T_+ increases but T_- decreases when A increases, and therefore we should consider the expression $A^{1/2}[t(Ar/\lambda)^{-1} - s(Ar/\lambda)^{-1}]$. The derivative of this expression with respect to A is equal to

$$2^{-1}A^{-1/2}t^{-3}s^{-3}(s-t)[s^2t^2+y(s^2+t^2+st-2st^2-2s^2t)],$$

where s and t are taken at $y = Ar/\lambda$. We proceed to show that the expression in the square brackets is positive. We observe first that $s < -t$ for $t > 0$ (this follows from the inequality $h(-t) > h(t)$) and

$$M = s^2+t^2+st-2st^2-2s^2t$$

$$= (1-2t)[s(s+t)+t^2/(1-2t)] > 0 \quad \text{for } 0 < t < 1/2.$$

Thus we can consider $s^2t^2M^{-1}+y$ and we show that this expression increases when t increases ($0 < t < 1/2$) and remains positive since the limit is zero when t tends to zero. We have

$$\frac{d(s^2t^2M^{-1}+y)}{dt} = -st^2M^{-2}\{3s(s+2t)-2s[(s+t)^2+2st]+$$

$$+t^2(3-2t)/(1-2t)\},$$

since

$$\frac{dt}{dt} = 1, \quad \frac{ds}{dt} = \frac{ds(h(t))}{dt} = ts^{-1}(1-2s)(1-2t)^{-1}.$$

If $s+2t < 0$ and $(s+t)^2+2st > 0$, i.e., for $\alpha = -t/s < \min(1/2, 2-3^{1/2}) = 2-3^{1/2}$, then this derivative is obviously positive. We notice that the expression in the brackets $\{\cdot\}$ is greater than

$$3s(s+2t)-4s^2t+t^2(3-2t)/(1-2t)$$

$$= s^2(1-2t)^{-1}[3(1-\alpha)^2+2\alpha(1-\alpha)(5-\alpha)s+8\alpha^2s^2],$$

where $0 < \alpha = -t/s < 1$. The discriminant $4\alpha^2(1-\alpha)^2(\alpha^2-10\alpha+1)$ of the quadratic expression in brackets is negative for $\alpha > 5-24^{1/2}$, and therefore this expression is positive. Our estimates exhaust all the cases since $5-24^{1/2} < 2-3^{1/2}$. This completes the proof of (b) and the proof of the Theorem.

Remark. If we notice that $T_+ + T_-$ is a continuous function,

$$\lim_{A \to 0} [T_+(A, \lambda) + T_-(A, \lambda)] = \pi\lambda^{-1/2}$$

(a general fact) and

$$\lim_{A \to \infty} T_+(A, \lambda) = \infty > \pi/2$$

(by (14)), then we can easily obtain an alternative proof of the existence of solutions of (10), (11).

The case $\nu = 0$. At the end we will show that the behaviour of the solutions of (1), (2) in the absence of viscosity ($\nu = 0$, see (6), (7)) is quite different from that for $\nu > 0$. Multiplying (7) by v and integrating on $[0, \pi]$ we obtain

$$\dot{z}/2 = Uz, \quad \text{where } z(t) = \int_0^\pi v^2(x, t)\,dx.$$

This together with (6) gives $\dot{U} = p - z$, $\dot{z} = 2Uz$ or $\ddot{U} - 2U\dot{U} + 2pU = 0$, which is of the same type as equation (11). We can apply results concerning the orbits of the vector field of (11) to see that $U \to \infty$ when $t \to \infty$ or (U, z) are periodic functions. The first case, where $U(t) = U_0 + pt$, $v \equiv 0$, shows what is possible in the absence of dissipation of energy: an unbounded increase of energy as the result of the work of constant exterior forces. The second case is not possible from the physical point of view.

We do not know whether the solution (U, v) of (6), (7) exists for all $t > 0$ and for arbitrary initial data nor whether it is unique (**P 1315**). The problem is analogous to that for degenerated Burgers–Hopf equation ((7) with $U \equiv 0$) studied by Hopf [7]. The preceding result on the periodic evolution of (U, z) (proved by the assumption that z is differentiable) and some numerical experiments suggest that such a solution may be not regular.

Acknowledgment. I am very grateful to Professor Andrzej Krzywicki for suggesting the problem and valuable discussions.

REFERENCES

[1] J. M. Burgers, *Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion*, Verhandlingen van de Koninklijke Nederlandsche Akademie van Wetenschappen, Amsterdam, Afd. Natuurkunde, 17 (2) (1939), p. 1–53.

[2] – *A mathematical model illustrating the theory of turbulence*, Advances in Applied Mechanics 1 (1948), p. 171–199.

[3] J. Cole, *On a quasi-linear parabolic equation occurring in aerodynamics*, Quarterly of Applied Mathematics 9 (3) (1951), p. 225–236.

[4] T. Dłotko, *The one-dimensional Burgers' equation: existence, uniqueness and stability,*

Zeszyty Naukowe Uniwersytetu Jagiellońskiego, Prace Matematyczne, 23 (1982), p. 157–172.

[5] — *The classical solution of the one-dimensional Burgers' equation*, ibidem 23 (1982), p. 173–182.

[6] — *Some remarks concerning the unstability of the one-dimensional Burgers' equation*, Annales Polonici Mathematici (to appear).

[7] E. Hopf, *The partial differential equation* $u_t + uu_x = \mu u_{xx}$, Communications on Pure and Applied Mathematics 3 (1950), p. 201–230.

[8] C. O. Horgan and W. E. Olmstead, *Stability and uniqueness for a turbulence model of Burgers*, Quarterly of Applied Mathematics 36 (2) (1978), p. 121–128.

[9] L. Nirenberg, *Topics in nonlinear functional analysis*, Courant Institute, New York 1974.

[10] D. H. Sattinger, *Topics in stability and bifurcation theory*, Lecture Notes in Mathematics 309, Springer Verlag 1973.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF WROCŁAW

Reçu par la Rédaction le 9.6.1982