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ON LCA GROUPS EACH OF WHOSE CHARACTERS IS ULTIMATELY MEASURABLE

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This paper is based on a result in Hewitt and Ross [3]. First, we recall a few definitions from [3] in a slightly different notation. Let G be an LCA group, $\hat{G}$ its group of continuous characters and $b\hat{G}$ the Bohr compactification of $\hat{G}$. Let λ be the Haar measure on G. By an invariant extension of λ we mean a countably additive, translation and inversion invariant nonnegative set function λ^* defined on a σ-algebra of subsets of G, containing all λ-measurable subsets (in the sense of [2]), and closed under translation and inversion of sets. Then a character ψ on G is said to be *ultimately measurable* if there is an invariant extension λ^* of λ such that every character in the subgroup of $b\hat{G}$ generated by ψ and $\hat{G}$ is λ^*-measurable. It was shown in [3], Theorem 3.13, that if $T(G)$, the torsion subgroup of G, is open, then every character of G is ultimately measurable. In this note we show that this sufficient condition is also necessary.

THEOREM. *Let G be an LCA group such that every character of G is ultimately measurable. Then $T(G)$ must be open.*

Proof. Assuming that $T(G)$ is not open, we shall show that G contains a proper dense subgroup H such that G/H is divisible. If G is not totally disconnected, G has a quotient group topologically isomorphic with the circle group T. As T has divisible quotient groups modulo proper dense subgroups, it follows via the continuous, open natural homomorphism of G onto T that G has the same property. Next, suppose G is totally disconnected. Let $E(G)$ denote the minimal divisible extension of G topologized in the usual manner so that G is an open subgroup. As $T(G)$ is nonopen, G contains a compact element of infinite order. Hence, $E(G)$ contains a copy of the rational group Q with some nondiscrete topology. Now $\bar{Q}$ is divisible ([1], Theorem 2), so that $E(G) = \bar{Q} + K$, for some subgroup K. Then $(Q+K) \cap G$ is a proper dense subgroup of G and the group $G/((Q+K) \cap G)$ is algebraically isomorphic with $(G+Q+K)/(Q+K)$

$= E(G)/(Q+K)$ which is evidently nontrivial and divisible. Thus G has in any case a proper dense subgroup H such that G/H is divisible, which we can assume to be a countable infinite divisible subgroup of T. This gives us a discontinuous character ψ of G with kernel H. Let n be a nonzero integer. As $\psi(G)$ is divisible, it is clear that the subgroup $\{g \in G\colon ng \in H\}$, the kernel of the character $n\psi$, is a proper dense subgroup of G. Hence $n\psi$ is discontinuous. Since the Haar measure of $\psi(G)$ is zero, we conclude that ψ is not ultimately measurable ([3], Theorem 3.9). The proof is now complete.

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