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# ON SOME CHARACTERIZATIONS OF THE COMPLEX NUMBER FIELD

BY

W. WIĘSŁAW (WROCŁAW)

**1.** In this note we shall give some characterizations of the complex number field. Recall some definitions. A field topology $\mathcal{T}$ is said to be *locally bounded* provided there exists a bounded neighbourhood $A$ of zero, i. e. for every neighbourhood $U$ of zero there exists another one $V$ such that $AV \subset U$. For any topological field $F$ we write $G(F)$ for the group of its all continuous automorphisms.

It is well known that the complex number field $\boldsymbol{C}$ has only two continuous automorphisms: $z \to z$ and $z \to \bar{z}$ (all other automorphisms of $\boldsymbol{C}$ are non-measurable). Moreover, $\boldsymbol{C}$ is complete in the usual product topology induced from $\boldsymbol{R}$ and is algebraically closed. It turns out that $\boldsymbol{C}$ is determined by these properties. In Theorem 1 which puts together various known and gives also some new results, we show that if $E$ is a locally bounded topological field which is algebraically closed and complete and group $G(E)$ is torsion, then $E = \boldsymbol{C}$. This result seems to be new although we use in the proof some rather known facts. Mutylin [11] proved in 1968 that if $E$ is a topological, locally bounded, complete and connected field, then $E = \boldsymbol{R}$ or $E = \boldsymbol{C}$. We present here another proof of this theorem (see Theorem 2). Our next result shows that an algebraically closed field $E$ of cardinality at least continuum has so many automorphisms as subsets. This generalizes a result of Soundararajan [12]. Finally, we give an example to show that the assumption "$G$ is torsion" is essential.

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**2.** We start with the following

THEOREM 1. *Let $E$ be a topological field provided with a non-discrete topology. Then the following conditions are equivalent:*

(1) *$E$ is topologically isomorphic to the complex number field $\boldsymbol{C}$,*

(2) *$E$ is a proper, locally bounded extension of the real number field $\boldsymbol{R}$,*

(3) *$E$ is locally bounded, complete and algebraically closed with $G(E)$ finite and non-trivial,*

(4) *$E$ is locally bounded, complete and algebraically closed with $G(E)$ torsion and non-trivial,*

(5) *$E$ is locally bounded, complete and connected with a non-trivial automorphism,*

(6) *$E$ satisfies the first axiom of countability* (i. e., *$E$ is metrizable*) *and $E$ is of inductive dimension 2.*

Proof. Equivalence of (1) and (2) was proved in [10], equivalence (1) $\Leftrightarrow$ (6) was shown in [1] and implication (1) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). Let $E$ be an algebraically closed field complete in a locally bounded field topology $\mathcal{T}$. Then ([8], Section 6) $\mathcal{T}$ cannot be non-trivially weakened (i. e., $\mathcal{T}$ is an atom in the lattice of all topologies on $E$). Let $F$ be the fixed field of $G(E)$. The Galois theory implies that $[E:F] =$ order $G(E) < \infty$.

Recall now a classical result (see, e. g., [6], Theorem 17, p. 318):

LEMMA 1. *Let $L$ be an algebraically closed field and $K \subset L$ its subfield. If $[L:K]$ is finite, then $K$ is really closed and $L = K(i)$, $i^2 = -1$.*

From Lemma 1 it follows that $E = F(i)$, where $F$ is really closed. The next lemma shows that $F$ is closed in $E$ and so it has to be complete in the topology $\mathcal{T}_1 = \mathcal{T} \mid F$.

LEMMA 2. *Let $E$ be a complete topological field such that $G = G(E)$ is finite. If $F$ is the field of invariants of $G$, then $F$ is closed and the topology $\mathcal{T}$ in $E$ is induced by the product topology $F$.*

Proof. Let $G = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$. We have to show that for every net $z_a$ in $E$, $z_a = x_1^{(a)} w_1 + x_2^{(a)} w_2 + \ldots + x_n^{(a)} w_n \xrightarrow{a} 0$ implies $x_j^{(a)} \xrightarrow{a} 0$ for all $j = 1, 2, \ldots, n$ (where $w_1, w_2, \ldots, w_n$ denotes a fixed basis of $E$ over $F$). From the definition of $G$ and from the continuity of $\varphi_1, \ldots, \varphi_n$ it follows that $\varphi_j(z_a) = x_1^{(a)} \varphi_j(w_1) + \ldots + x_n^{(a)} \varphi_j(w_n) \xrightarrow{a} 0$.

First we will show that

$$\det\big(\varphi_i(w_j)\big) \neq 0. \tag{i}$$

In fact, if $\det\big(\varphi_i(w_j)\big) = 0$, then one of the rows of the matrix $\big(\varphi_i(w_j)\big)$ is a linear combination of all others. This gives

$$\varphi_k(w_s) = \sum_{j \neq k} d_j \varphi_j(w_s) \tag{ii}$$

for some $1 \leqslant k \leqslant n$ and all $s = 1, 2, \ldots, n$, where $d_1, d_2, \ldots, d_n$ belong to $E$ and do not all vanish. Since every element $x \in E$ has the form $x_1 w_1 + x_2 w_2 + \ldots + x_n w_n$, where $x_1, x_2, \ldots, x_n \in F$, we obtain from (ii)

$$\varphi_k(x) = \sum_{j \neq k} d_j \varphi_j(x) \tag{iii}$$

for every $x \in E$, contradicting the linear independence of automorphisms (see [6], Chapter I, Theorem 3). Hence (i) is true. From the Cramer formulae we obtain

$$x_j^{(a)} = \lambda_1^{(j)}\varphi_1(z_a) + \ldots + \lambda_n^{(j)}\varphi_n(z_a) \tag{iv}$$

for $j = 1, 2, \ldots, n$. By the continuity of $\varphi_j$ it follows that $x_j^{(a)} \xrightarrow{a} 0$ $(j = 1, 2, \ldots, n)$.

Therefore $F$ is closed in $E$: if $x_a w_1 \xrightarrow{a} a \in E$, $a = a_1 w_1 + \ldots + a_n w_n$, then $x_a \xrightarrow{a} a_1$, $a_k = 0$ for $k \neq 1$. Completeness of $F$ is now obvious.

In order to prove our theorem, we will need yet the following lemma proved by Kowalsky and Dürbaum [9]:

LEMMA 3. *Every locally bounded full topology (i. e., topology of a field which cannot be non-trivially weakened) in the field $Q$ of rational numbers is equivalent to a topology induced by a valuation.*

By the theorem of Ostrowski and by Lemma 3 our topology $\mathcal{T}$ is determined on $Q$ by a real absolute value or by a $p$-adic valuation. Thus $F$ must contain topologically either $\boldsymbol{R}$ or $Q_p$. The last case is impossible, since it would imply that $Q_p$ is formally real, and this is not true, because every form $x_1^2 + x_2^2 + \ldots + x_r^2$ has a non-trivial zero in $Q_p$ for every $r \geqslant 5$ (see [2], Theorem 5, p. 74). Therefore, $F$ contains topologically $\boldsymbol{R}$. Since the order in $\boldsymbol{R}$ is unique, the order in $F$ restricted to $\boldsymbol{R}$ coincides with the usual ordering of $\boldsymbol{R}$. Observe that $F$ has no non-trivial continuous automorphisms. Consequently, $\boldsymbol{R} = F$, as the only proper locally bounded extension of $\boldsymbol{R}$ is $\boldsymbol{C}$ (see [10]). It follows now from Lemma 1 that $E$ is topologically isomorphic to $\boldsymbol{C}$, and so the implication (3) $\Rightarrow$ (1) follows.

Obviously (3) implies (4). To prove the converse implication assume that (4) holds. Let $H$ be a finite cyclic group generated by a suitably chosen non-trivial continuous automorphism of $E$. Now we can repeat the arguments we used proving (3) $\Rightarrow$ (1), taking for $F$ the field of invariants of $H$ to get (3). It gives equivalence of (4) and (1).

Equivalence of (1) and (5) was originally proved by Mutylin [11]; we will present here another proof. It will follow from a more general theorem. (For a further generalization, see [15].)

THEOREM 2. *Let $E$ be a field provided with a locally bounded topology. If $E$ is connected and complete in this topology, then it is topologically isomorphic either to the real number field $\boldsymbol{R}$ or to the complex number field $\boldsymbol{C}$.*

Proof of this theorem will be based on three lemmas.

LEMMA 4. *Let $E$ be a locally bounded topological field, connected in this topology. Then $E$ has a non-zero topological nilpotent (i. e., an element $x \neq 0$ such that $x^n \to 0$).*

Proof. We use the well-known fact that if $G$ is a connected group, then an arbitrary neighbourhood of the unit element generates $G$.

Let $U$ be an arbitrary, symmetrical neighbourhood of zero in $E$. Put

$$n\circ U = U+U+\ldots+U \quad (n \text{ times}).$$

Obviously,

$$E = \bigcup_{n=1}^{\infty} n\circ U.$$

Since topology in $E$ is locally bounded, there is a non-zero element $x$ in $E$ such that $xU+xU \subset U$ or $2\circ U \subset x^{-1}U$.

By a simple induction we obtain $2^k\circ U \subset x^{-k}U$ for all $k = 1, 2, 3, \ldots$ Since $n\circ U \subset m\circ U$ for $n \leqslant m$, we have

$$E = \bigcup_{k=0}^{\infty} 2^k\circ U = \bigcup_{k=0}^{\infty} x^{-k}U.$$

We will show that $x$ is a topological nilpotent. This is equivalent to

$$\bigcap_{n=1}^{\infty} x^n U = \{0\}.$$

Indeed, for otherwise there would be in $E$ an $x_0 \neq 0$ such that $x_0 \in x^n U$ for $n = 1, 2, 3, \ldots$ and this would imply $x^{-n} \in x_0^{-1}U$, i. e.,

$$E = \bigcup_{n=0}^{\infty} x^{-n}U = x_0^{-1}U^2, \quad E = U^2,$$

which is impossible for sufficiently small $U$.

LEMMA 5. *Let $E$ be a field provided with a locally bounded topology. If the set $T$ of all topological nilpotents contains a non-zero element, then it is open.*

Proof. Let $0 \neq x \in T$ and let $W$ denote a bounded neighbourhood of zero in $E$. First, we will prove that there is a neighbourhood $U$ of zero such that for every neighbourhood $V$ there is an $n_0$ such that $U^n \subset V$ for all $n \geqslant n_0$.

Let $U$ be any neighbourhood satisfying $WU \subset xW$ and let $V$ be an arbitrary neighbourhood of zero in $E$. Then $x^n W \subset V$ for all $n \geqslant n_0$. Moreover, $V \supset x^n W = x^{n-1}(xW) \supset x^{n-1}UW = x^{n-2}(xW)U \supset x^{n-2}U^2 \supset \ldots \supset U^n$ for all $n \geqslant n_0$.

Thus we have found an open set $U$ such that

(a) if $x \in T$, then, for some $N \geqslant 1$, $x^N \in U$,

(b) $U \subset T$.

Observe that if, for some $x$, $x^N \in U$, then $x \in T$. Indeed, in view of (b), the sequence $x^N, x^{2N}, x^{3N}, \ldots$ tends to zero, and — by the continuity of multiplication — each of the sequences $x^{kN+j} = x^j x^{kN}$ ($k = 1, 2, 3, \ldots$; $j = 1, 2, \ldots, N-1$) also tends to zero, thus $x \in T$ as asserted.

Concluding, we get

$$T = \bigcup_{N=1}^{\infty} \{x\colon x^N \in U\}$$

and, since $x \to x^N$ is continuous, and $U$ is open, we get the openess of $T$.

LEMMA 6 (see [13]). *Let $E$ be a topological field. Then the topology in $E$ can be normed (induced by a real valuation) if and only if the set $T$ of topological nilpotents in $E$ is open and $(E\setminus T)^{-1}$ is bounded.*

Proof of Theorem 2. $E$ is locally bounded, complete and connected. Lemma 4 implies existence of topological nilpotents in it. By Lemma 5 the set $T$ is open. But our topology is full and locally bounded, because $E$ is complete in a locally bounded topology, and so $(E\setminus T)^{-1}$ is bounded. From Lemma 6 it follows that $E$ is a field with a real valuation. We observe now that $E$ is of characteristics 0, since otherwise the valuation would be a non-archimedean which contradicts connectedness of $E$ (see [5]). Hence $E$ is an archimedean complete field, and so $E = \boldsymbol{R}$ or $E = \boldsymbol{C}$. Theorem 2 is proved. (This theorem generalizes a well-known theorem of Pontrjagin on connected locally compact fields.)

From Theorem 2 it follows that (1) is equivalent to (5).

The proof of Theorem 1 is complete.

**3.** In connection with Theorem 2 let us recall that Dieudonné [3] has given an example of a connected subfield of $\boldsymbol{C}$ different from $\boldsymbol{R}$ and $\boldsymbol{C}$ and Kapuano [7] has shown the existence of a one-dimensional subfield $L$ of $\boldsymbol{C}$, $L \neq \boldsymbol{R}$. This furnishes a counter-example to a conjecture of Baer and Hasse [1] that $\boldsymbol{R}$ is the only one-dimensional subfield of $\boldsymbol{C}$. From Theorem 2 it follows that the only locally bounded complete, connected and one-dimensional field is the real number field $\boldsymbol{R}$.

CONJECTURE. $\boldsymbol{R}$ is the only one-dimensional complete and connected topological field. (**P 762**)

In connection with Theorem 1 let us notice th t non-discreteness of topology is essential. In fact, every (discrete) algebraically closed field has a large number of automorphisms as shown in the following

THEOREM 3. *Every algebraically closed field of cardinality $|E| \geqslant c$ has $2^{|E|}$ automorphisms.*

Proof. In order to prove the theorem it is sufficient to show that $E$ has at least $2^{|E|}$ automorphisms. Let $P$ be the prime field of $E$ and let $B$ denote the transcendental basis of $E$ over $P$. Then $E$ is an algebraic extension of $P(B)$. Let $|E| = |B| = \alpha$ and let $\varphi$ be an arbitrarily chosen permutation of $B$. We extend $\varphi$ in an obvious way to an automorphism $\bar{\varphi}$ of $P(B)$: let $a = f(b_1, b_2, \ldots, b_n) \in P(B)$, where $f$ is a rational function with coefficients in $P$.

Put

$$\bar{\varphi}(f(b_1, b_2, \ldots, b_n)) = f(\varphi(b_1), \varphi(b_2), \ldots, \varphi(b_n)).$$

Now take $f(x) \in K[x]$, where $K = P(B)$, and assume that $f(x)$ is irreducible over $K$. Then $\bar{f}(x) = \bar{\varphi}(f(x))$ is again irreducible over $K$. Let $\xi_1, \ldots, \xi_n$ be all roots of $f$ in $E$ and similarly denote by $\eta_1, \ldots, \eta_n$ all roots of $\bar{f}$. Hence the mapping $\bar{\varphi}_f$ defined by $\bar{\varphi}_f(\xi_k) = \eta_k$ $(k = 1, 2, \ldots, n)$ can be extended to the isomorphism $\varphi_f$ of splitting fields of $f$ and $\bar{f}$ over $K$, respectively. Let now $\{f_a\}$ be the well ordered set of all irreducible polynomials over $K$. Denote by $K_a$ the composite of all splitting fields of the polynomials $f_\beta$ for $\beta < a$ and suppose that the isomorphism $\varphi_a: K_a \to \bar{K}_a$ has been established. As before extend it to the isomorphism $\bar{\varphi}_a : K_a(f_a) \to \bar{K}_a(\bar{f}_a)$, where $K_a(f_a)$ is the splitting field of $f_a$ over $K_a$. By the Kuratowski-Zorn lemma, isomorphism $\bar{\varphi}_a$ can be extended to an automorphism $\tilde{\varphi}$ of $E$. Now it remains to note that two different permutations $\varphi \neq \psi$ give different automorphisms of $E$, $\tilde{\varphi} \neq \tilde{\psi}$. This proves the theorem.

**4.** Let us remark that assumption (4) in Theorem 1 that the group $G(E)$ of all continuous automorphisms of $E$ is torsion, is essential. In fact, let $F$ be a non-archimedean ordered field, complete in the topoolgy induced by that order. Then $F = K(B)$, where $K$ is the maximal archimedean ordered subfield of $F$ and $B$ is the set of transcendental elements over $K$ (see [4], Theorem 4). We can take $B$ to be the transcendental base of extension $F$ over $K$. Let $B = \{b_a\}$. Obviously, since the order is non-archimedean and $b_a \notin K$, we have $b_a > q$ for every $q \in Q$ and $b_a > k$ for all $a$ and for every $k \in K$.

We put $F_1 = K(b)$, where $b$ is any arbitrarily fixed element of $B$. The mapping $b \to b+1$ can be extended to an automorphism $\varphi_1$ of $F_1$ by putting

$$\varphi_1 |_K = id_K \quad \text{and} \quad \varphi_1(f(b)/g(b)) = f(b+1)/g(b+1),$$

where $f$ and $g$ are polynomials over $K$.

Moreover, let us note that the inequality

$$\frac{a_n b^n + \ldots + a_0}{c_m b^m + \ldots + c_0} > 0 \quad \text{for } a_i, c_j \in K,$$

holds if and only if $a_n c_m > 0$.

Using the Kuratowski-Zorn lemma, we can extend $\varphi_1$ to an automorphism $\varphi$ of $F$. Since $\varphi_1$ preserves the order, it is continuous. Hence $\varphi$ is continuous. We extend $\varphi$ to a continuous automorphism $\tilde{\varphi}$ of $E = F(i)$ by the formula

$$\tilde{\varphi}(a+bi) = \varphi(a) + i\varphi(b), \quad a, b \in F.$$

Hence $\tilde{\varphi}$ is a continuous automorphism of $E = F(i)$, $i^2 = -1$. Obviously, $\tilde{\varphi}$ has infinite order in the group $G(E)$.

On the other hand, the local boundedness in Theorem 1 ((3) and (4)) of the field topology seems to be a technical condition and I believe that the theorem remains true without that assumption. (**P 763**)

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INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY

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