COLLOQUIUM MATHEMATICUM

VOL. XXXIX 1978 FASC. 1

# COMPACT HAUSDORFF SPACES WITH TWO OPEN SETS

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Schoenfeld and Gruenhage [5] have shown that if a compact metric infinite space $X$ has, up to a homeomorphism, only two open non-empty subsets, then it is homeomorphic to the Cantor set. To show this they have proved that

(1) $X$ does not contain isolated points,

(2) $X$ is totally disconnected.

The aim of this note is to discuss the compact Hausdorff infinite spaces having, up to a homeomorphism, only two open non-empty subsets. For the brevity, we shall say that these (infinite) spaces have *two open sets*.

We show that if $X$ is a compact Hausdorff space with two open sets, then

(3) $X$ has the Souslin property hereditarily,

(4) $X$ has a countable base around each closed (non-open) subset still having properties (1) and (2).

A known space, called by some authors the *double arrow*, described in *Mémoire* by Alexandroff and Urysohn, 1929, is an example of a compact Hausdorff separable non-metrizable space with two open sets.

If the space $X$ is not separable, then it is nowhere separable and we show that

(5) $X$ is a union of an increasing sequence of $\aleph_1$ nowhere dense separable closed subsets.

In consequence,

(6) $X$ contains a dense subset of cardinality $\aleph_1$,

(7) $X$ has cardinality of continuum.

Clearly, such a space cannot be constructed within ZFC, since its existence contradicts the axioms of the theory ZFC $+(2^{\aleph_0} > \aleph_1)+$ Martin's Axiom. However, if there exist homogeneous Souslin lines, then compact Hausdorff non-separable spaces with two open sets can be constructed in a similar way as in the separable case.

All compact Hausdorff spaces with two open sets which we have known so far are ordered. Do there exist non-ordered ones? **(P 1027)**

References to consistency problems concerning axioms of set theory can be found in [2] and [1].

In the first two sections, $X$ will denote a given compact Hausdorff space with two open sets.

**1. Spaces with two open sets, in general.** One of these open sets is compact, being homeomorphic to the whole space $X$. The second one is non-compact, being homeomorphic to each of the subspaces $X-\{x\}$. This follows from the fact that

(1) $X$ does not have isolated points, $X$ being infinite.

As in the metric case,

(2) $X$ is totally disconnected.

In the proof of (2) by Schoenfeld and Gruenhage the metric is involved but, as can be seen, the Hausdorff separation property is sufficient to perform the proof. Since the subspaces $X-\{x\}$ are all homeomorphic, we infer easily that $X$ is homogeneous.

**1.1.** *If $U$ is a non-compact open subset of $X$, then $U = U_1 \cup U_2 \cup \ldots$, where $U_i$ are mutually disjoint, closed-open, and non-empty.*

Proof. One can construct by induction a sequence $U_1, U_2, \ldots$ of closed-open, non-empty, and mutually disjoint subsets of $X$. The union $U_1 \cup U_2 \cup \ldots$ is open and non-compact. If $U$ is an arbitrary non-compact open subset of $X$, then $U$ is homeomorphic to that union.

**1.2.** COROLLARY. *If $F$ is a closed non-open subset of $X$, then there exists a countable base of open subsets around $F$; in particular, $X$ is first countable.*

Proof. The set $X-F$ is open and non-compact, whence, by 1.1,

$$X-F = U_1 \cup U_2 \cup \ldots,$$

where $U_i$ are non-empty, closed-open, and mutually disjoint. The sets

$$V_i = X-(U_1 \cup U_2 \cup \ldots \cup U_i)$$

form a base around $F$. In fact, if $W$ is a neighbourhood of $F$, we may assume that $W$ is closed-open. Then $U_i \subset W$ for all but finitely many $i$, for otherwise $X-W$ would be a union of infinitely many mutually disjoint closed-open sets, which would contradict the compactness of $X-W$. Thus $V_i \subset W$ for some $i$.

**1.3.** COROLLARY. *$X$ has the Souslin property hereditarily.*

Proof. Let $D$ be an infinite discrete subspace of $X$. For each $x$ from $D$ take an open subset $U(x)$ of $X$ such that $D \cap U(x) = \{x\}$. The open set $V = \bigcup \{U(x): x \in D\}$ is non-compact, $D$ being infinite. By 1.1, there exist closed-open, non-empty, and mutually disjoint subsets $V_1, V_2, \ldots$ of $X$ such that $V = V_1 \cup V_2 \cup \ldots$ It follows that $D \subset V$, and each $D \cap V_i$ is finite in view of compactness of $V_i$. Thus $D$ is countable.

Since each discrete subspace of $X$ is at most countable, each subspace of $X$ has the Souslin property.

Having 1.1 in view, for each closed and non-open subset $F$ of $X$ fix a countable base $H(F)$ of closed-open neighbourhoods around $F$. For each separable subset $F$ of $X$ fix a countable dense subset $D(F)$ of $F$. And let us fix a choice function $h$ assigning to each non-empty subset $A$ of $X$ a point $h(A)$ in $A$.

**2. The non-separable case.** If $X$ is not separable, then each non-empty closed-open subset of $X$ is not separable, being homeomorphic to $X$. Since closed-open subsets form a base, each separable subspace of $X$ is nowhere dense.

**2.1.** *If $X$ is not separable, then*

$$X = \bigcup \{F_a\colon a < \omega_1\},$$

*where $F_a$ are closed separable subsets of $X$ such that $F_a$ is a nowhere dense subset of $F_\beta$ provided $a < \beta$.*

Proof. We construct the sets $F_a$ by induction. Let $F_0 = \emptyset$. Let $F_a = \mathrm{cl} \bigcup \{F_\beta\colon \beta < a\}$ if $a$ is a limit ordinal. To construct the set $F_{a+1}$ from the preceding ones, define $A_a$ to be the set of values of the choice function $h$ on sets $V - F_a$, where $V$ are in $H(\{x\})$ and $x$ are points of $D(F_a)$, and on sets $X - W_1 \cup W_2 \cup \ldots \cup W_k$ for each finite subfamily $\{W_1, W_2, \ldots$ $\ldots, W_k\}$ of the family $\bigcup \{H(F_\beta)\colon \beta \leqslant a\}$. The set $A_a$ is countable, $D(F_a)$ and $a$ being countable. Let $F_{a+1} = F_a \cup \mathrm{cl} A_a$.

Clearly, $F_a$ is separable if $F_\beta$, $\beta < a$, is separable, and $F_a$ is a nowhere dense subset of $F_{a+1}$, since each $x$ in $D(F_a)$ is a limit point of $A_a$, and so is each other point of $F_a$, $D(F_a)$ being dense in $F_a$ and $F_a$ being first countable. Clearly, $F_a$ are closed by the construction.

The union $S = \bigcup \{F_a\colon a < \omega_1\}$ is a closed subset of $X$. In fact, let $x$ be an accumulation point of $S$. Take a countable base $B$ of open neighbourhoods of $x$ (the existence is assured by 1.2). We have $S \cap V \neq \emptyset$ for $V$ from $B$. This means that for each $V$ from $B$ there is an $a_V$, $a_V < \omega_1$, such that $V \cap F_{a_V} \neq \emptyset$. Let $a$, $a < \omega_1$, be greater than each $a_V$. We have $V \cap F_a \neq \emptyset$ for each $V$, $F_a$ containing each $F_{a_V}$. Hence $x \in F_a$, $F_a$ being closed. Thus $x \in S$.

We claim that $S = X$. To show this suppose that there exists a point $p$ in $X - S$. For each $a$ take a closed-open neighbourhood $V_a$ of $F_a$, being a member of $H(F_a)$, such that $p \notin V_a$. It follows that $S \subset \bigcup \{V_a\colon a < \omega_1\}$ and, in view of compactness of $S$, a finite number of $V_a$'s, say $V_{a_1}, V_{a_2}, \ldots$ $\ldots, V_{a_k}$, cover $S$. We have $V_{a_1} \cup V_{a_2} \cup \ldots \cup V_{a_k} \neq X$, since the sets $V_a$ do not contain $p$. Let $a$, $a < \omega_1$, be greater than each of $a_1, a_2, \ldots, a_k$. By the construction, there is a point of $F_{a+1}$, and so a point of $S$, in the set $X - V_{a_1} \cup V_{a_2} \cup \ldots \cup V_{a_k}$. A contradiction with $S \subset V_{a_1} \cup V_{a_2} \cup \ldots \cup V_{a_k}$.

**2.2.** COROLLARIES. 1. *$X$ contains a dense subset of cardinality $\aleph_1$.* 2. *The cardinality of $X$ is that of continuum.*

The first follows from the fact that $X$ is a union of $\aleph_1$ separable subspaces. The second follows from the first, since each point of $X$ is a limit of a sequence of points from each arbitrarily given dense subset of $X$, $X$ being first countable.

The construction from 2.1 resembles constructions on Souslin lines known since long (see the book by Devlin and Johnsbråten [1], p. 13, as well as more recent papers, leading to estimations of cardinalities of spaces, by Ponomarev [4], Pol [3], and Šapirovskiĭ [6]).

The referee pointed out that for compact Hausdorff spaces $X$ without non-empty separable open subsets and having countable bases around closed non-open subsets (our spaces $X$ satisfy these conditions if they are non-separable) the conclusion of our Theorem 2.1 follows from the existence of a dense subset of $X$ having cardinality $\aleph_1$ (the proof by transfinite induction), and that the last property can be deduced from Theorem 1 or Corollary 1 of [6]. This is another way of obtaining our Theorem 2.1 from Corollary 1.2.

**3. Examples.** There is a general scheme for examples that we have known so far.

**3.1.** LEMMA. *If a compact ordered space $X$ has the Souslin property, and the closed-open intervals of $X$ form a base and are all homeomorphic to $X$, then compact open non-empty subsets of $X$ are all homeomorphic to $X$, and all non-compact open subsets of $X$ are homeomorphic each to other.*

Proof. Note first that if $V$ is a non-compact open interval of $X$, then in view of the Souslin property of $X$, $V$ is a union of countably many closed-open non-empty intervals mutually disjoint.

If $U$ is an open subset of $X$, then $U = U_1 \cup U_2 \cup \ldots$, where $U_i$ are maximal open intervals contained in $U$; this is assured by the Souslin property. The intervals are disjoint if they are different.

In the case where $U$ is compact, we have $U = U_1 \cup U_2 \cup \ldots \cup U_n$ for some $n$, and all $U_i$ are compact. Take an arbitrary decomposition of $X$ into $n$ disjoint closed-open non-empty intervals, $X = V_1 \cup V_2 \cup \ldots \cup V_n$. By the assumption, there is a homeomorphism from $U_i$ onto $V_i$ for each $i$ (all these intervals are homeomorphic to $X$). The union of these homeomorphisms is a homeomorphism between $U$ and $X$.

In the case where $U$ is non-compact, decompose each non-compact $U_i$ into countably many closed-open non-empty intervals, according to the remark made at the beginning of the proof. We have $U = W_1 \cup W_2 \cup \ldots$, where $W_i$ are closed-open non-empty intervals mutually disjoint. Having another non-compact open subset $U'$ of $X$ and taking a decomposition

$U' = W_1' \cup W_2' \cup \ldots$ as before, we get a homeomorphism between $U$ and $U'$, the union of homeomorphisms between $W_i$ and $W_i'$.

An ordered continuum is said to be (*order*) *homogeneous* if each two its closed intervals which do not reduce to points are similar. If $a$ and $b$ are the first and the last elements, respectively, in an ordered continuum, then it will be denoted by $[a, b]$. By $[x, y]$, where $a \leqslant x < y \leqslant b$, we denote closed intervals of $[a, b]$; the symbol $(x, y]$ stands for an interval without the end $x$. In the set $\{0, 1\}$ which we consider in the sequel an order given by $0 < 1$ is assumed.

**3.2.** LEMMA. *If $[a, b]$ is an ordered homogeneous continuum having the Souslin property, then the space*

$$X = (a, b] \times \{0\} \cup [a, b) \times \{1\}$$

*with topology given by the lexicographical order is a compact Hausdorff space with two open sets.*

Proof. Clearly, $X$ is compact. $X$ has the Souslin property, since $[a, b]$ has. Compact non-empty intervals are all of the form

$$U = (x, y] \times \{0\} \cup [x, y) \times \{1\}.$$

Each homeomorphism from $[a, b]$ onto $[x, y]$ induces a homeomorphism from $X$ onto $U$. These compact open intervals form a base of $X$. By Lemma 3.1, $X$ has only two open sets.

Example 1. Let $[a, b]$ be a closed interval of the reals. The space $X = (a, b] \times \{0\} \cup [a, b) \times \{1\}$ (this space is called the *double arrow*) defined as in Lemma 3.2 is a separable non-metrizable compact Hausdorff space having, by Lemma 3.2, two open sets; the space is non-metrizable containing copies, $(a, b] \times \{0\}$ and $[a, b) \times \{1\}$, of Sorgenfrey (half-) lines.

Example 2. Let $[a, b]$ be an ordered homogeneous continuum which is not separable and which has the Souslin property. Such continua, Souslin homogeneous continua, exist in the theory ZFC+Axiom of Constructibility (see [1], p. 40, where the existence of homogeneous Souslin continua is proved even under much weaker assumption known as the $\diamondsuit$-hypothesis). The space formed from $[a, b]$ as in Lemma 3.2 is a compact Hausdorff non-separable space with two open sets.

The problem to find all compact Hausdorff spaces with two open sets seems to be open, even the problem to find all such spaces among separable ones. (**P 1028**)

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*Reçu par la Rédaction le 20. 4. 1976*