COLLOQUIUM MATHEMATICUM

VOL. XXII 1971 FASC. 2

REMARKS ON ALGEBRAS HAVING TWO BASES OF DIFFERENT CARDINALITIES

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Let $\mathfrak{A} = (X; F)$ be an abstract algebra and let $S(\mathfrak{A})$ denote the set of all n such that in $\mathfrak{A}$ there exists an essentially n-ary algebraic operation, i.e, an operation depending on all its variables.

E. Marczewski has raised the following conjecture (see [1]): if $\mathfrak{A}$ contains two bases of different cardinalities, then $S(\mathfrak{A})$ contains all positive n (for the definition of bases see [1]). Observe that because of the existence of the trivial unary operation $e_1^1(x) = x$ there is $1 \in S(\mathfrak{A})$ for arbitrary $\mathfrak{A}$.

Narkiewicz [2] obtained some partial results connected with the conjecture. In particular, he proved that

(i) if $\mathfrak{A}$ contains two bases of different cardinalities, then $2 \in S(\mathfrak{A})$.

In this paper we prove some further results (Theorems 1, 2 and 3).

If $\mathfrak{A} = (X; F)$ is an abstract algebra, then by $I(\mathfrak{A})$ we denote an algebra $(X;\ I(F))$, where $I(F)$ is the set of all idempotent algebraic operations $f(x_1, \ldots, x_n)$, i.e., of all operations satisfying equality $f(x, x, \ldots, x) = x$. The algebra $I(\mathfrak{A})$ is called the *maximal idempotent reduct* of $\mathfrak{A}$.

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ be two bases of $\mathfrak{A}$ such that $m < n < \aleph_0$. It is easy to check that

$$\text{(ii)}\qquad \begin{aligned} f_i(g_1, g_2, \ldots, g_n)(x_1, x_2, \ldots, x_m) &= x_i \quad (i = 1, 2, \ldots, m),\\ g_j(f_1, f_2, \ldots, f_m)(y_1, y_2, \ldots, y_n) &= y_j \quad (j = 1, 2, \ldots, n),\end{aligned}$$

where f_i and g_j are some algebraic operations in $\mathfrak{A}$.

THEOREM 1. *If $\mathfrak{A}$ contains two bases of different cardinalities, then the set $S\big(I(\mathfrak{A})\big)$ is infinite.*

Proof. Consider the operations

$$\begin{aligned} F_i &= F_i(x_1^1, x_2^1, \ldots, x_m^1, x_1^2, x_2^2, \ldots, x_m^2, \ldots, x_1^n, x_2^n, \ldots, x_m^n)\\ &= f_i\big(g_1(x_1^1, x_2^1, \ldots, x_m^1), g_2(x_1^2, x_2^2, \ldots, x_m^2), \ldots, g_n(x_1^n, x_2^n, \ldots, x_m^n)\big),\end{aligned}$$

where f_i and g_j satisfy (ii). Obviously, all F_i are idempotent. Observe that for every $k = 1, 2, \ldots, n$ there exists an operation F_i depending on some variable $x_{i_0}^k$, where $1 \leqslant i_0 \leqslant m$, for otherwise there would exist k_0 such that no operation F_i would depend on $x_1^{k_0}, x_2^{k_0}, \ldots, x_m^{k_0}$, and hence the operation

$$\begin{aligned} g_{k_0}&(x_1^{k_0}, x_2^{k_0}, \ldots, x_m^{k_0}) \\ &= g_{k_0}(F_1, F_2, \ldots, F_m)(x_1^1, \ldots, x_m^1, x_1^2, \ldots, x_m^2, \ldots, x_1^{k_0}, \ldots, x_m^{k_0}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \ldots, x_1^n, \ldots, x_m^n) \end{aligned}$$

would be an algebraic constant, a contradiction with (ii).

Now we shall prove that among operations F_i there exists one depending on p variables, where $p \geqslant n/m$. In fact, if each F_i depends on less than n/m variables, then the set of variables on which the operations F_i depend will be of the cardinality less than $m(n/m) = n$ which gives a contradiction with the first part of the proof. Without loss of generality we may assume that m is the minimal cardinality of bases in $\mathfrak{A}$ and n is the next one. By a theorem of Marczewski (see [1]) numbers of elements of bases in $\mathfrak{A}$ form arithmetical progress $l_s = m+sr$, where $s = 0, 1, \ldots$, $r = n-m$. Let q be a natural number. Then there exists a base B_s such that $|B_s| = l_s$ and $l_s/m \geqslant q$, and among the operations F_i defined for the bases A and B_s there exists an operation depending on at least q' variables, where $l_s \cdot m \geqslant q' \geqslant l_s/m$. Because q was arbitrary, we get the thesis of Theorem 1.

From Theorem 1 and results of K. Urbanik (see [3]) we get

COROLLARY. *If $\mathfrak{A}$ is an algebra with two bases of different cardinalities, then $S(I(\mathfrak{A}))$ is of one of the following forms*: $\{1, 3, 5, \ldots\}$, $\{m, m+1, \ldots\}$, $\{1, 2, 3, \ldots, n\} \cup \{m, m+1, \ldots\}$, $\{1, 3, 5, \ldots\} \cup \{m, m+1, \ldots\}$.

THEOREM 2. *If $\mathfrak{A}$ contains two bases of different cardinalities, then there exists n_0 such that $2 \in S(\mathfrak{A})$ and $n \in S(\mathfrak{A})$ for all $n \geqslant n_0$.*

Proof. $2 \in S(\mathfrak{A})$ by (i). Consider $S(I(\mathfrak{A}))$. In view of the Corollary it remains to check the case $S(I(\mathfrak{A})) = \{1, 3, 5, \ldots\}$. But from [3] (Theorem 2, part 3, p. 139) it follows that $I(\mathfrak{A})$ is then a maximal idempotent reduct of a Boolean group, i.e., of a group satisfying $2x = 0$. This reduct can be considered as the algebra $(X; x_1+x_2+x_3)$, where $+$ is the group operation. Let $x \cdot y$ be an essentially binary operation which exists by (i). Then the operation $(x_1+x_2+\ldots+x_{2n-1}) \cdot x_{2n}$ is essentially $2n$-ary. In fact, because $+$ is commutative, it depends on all variables $x_1, x_2, \ldots, x_{2n-1}$ or on none of them. If it does not depend on x_{2n} or if it depends on none of the remaining variables, then the identification $x_1 = x_2 = \ldots = x_{2n-1} = x$ gives a contradiction with the assumption that the operation $x \cdot x_{2n}$ is essentially binary. Hence $S(\mathfrak{A}) = \{1, 2, 3, \ldots\}$.

THEOREM 3. *If $\mathfrak{A}$ contains two bases of different cardinalities and does not contain any algebraic constant, then*

$$S(I(\mathfrak{A})) \supseteq \{1, 2, \ldots, k\} \cup \{l, l+1, \ldots\}$$

for some k, l $(2 \leqslant k \leqslant l)$.

Proof. Consider the operations

$$H_i = H_i(x_1, x_2, \ldots, x_n) = f_i(\hat{g}_1(x_1), \hat{g}_2(x_2), \ldots, \hat{g}_n(x_n)),$$

where $\hat{g}(x) = g(x, x, \ldots, x)$.

Take the substitution

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ i_1 & i_2 & \ldots & i_n \end{pmatrix}, \quad 1 \leqslant i_k \leqslant 2, \ k = 1, 2, \ldots, n,$$

and put

$$H_i(\sigma)(x_1, x_2) = H_i(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), \quad \text{where } i = 1, 2, \ldots, m.$$

We shall prove that among operations $H_i(\sigma)$ there exists an essentially binary one. The operations H_i are idempotent by (ii).

Suppose to the contrary that all $H_i(\sigma)$ are trivial. This means that $H_i(\sigma) = H_i(\sigma)(x_1, x_2) = x_{\varepsilon(\sigma, i)}$, where $\varepsilon(\sigma, i) \in \{1, 2\}$. Define the mapping

$$\varphi(\sigma) = (\varepsilon(\sigma, 1), \varepsilon(\sigma, 2), \ldots, \varepsilon(\sigma, m)).$$

Let $\varphi(\sigma_1) = \varphi(\sigma_2)$. Then $\varepsilon(\sigma_1, k) = \varepsilon(\sigma_2, k)$ for $k = 1, 2, \ldots, m$. Putting

$$\sigma_1 = \begin{pmatrix} 1 & 2 & \ldots & n \\ i_1 & i_2 & \ldots & i_n \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 1 & 2 & \ldots & n \\ j_1 & j_2 & \ldots & j_n \end{pmatrix},$$

we have

$$H_k(\sigma_1)(x_1, x_2) = H_k(\sigma_2)(x_1, x_2)$$

and

$$\begin{aligned} \hat{g}_k(x_{i_k}) &= g_k(H_1(\sigma_1)(x_1, x_2), \ldots, H_m(\sigma_1)(x_1, x_2)) \\ &= g_k(H_1(\sigma_2)(x_1, x_2), \ldots, H_m(\sigma_2)(x_1, x_2)) = \hat{g}_k(x_{j_k}). \end{aligned}$$

Hence $i_k = j_k$ for $k = 1, 2, \ldots, n$, because $\mathfrak{A}$ does not contain any algebraic constant. Thus we see that φ is one-to-one but it is impossible, because there does not exist a one-to-one mapping of the 2^n-element set into 2^m-element set. Thus we infer that in $I(\mathfrak{A})$ there exists an essentially binary operation and our theorem easily follows from the corollary.

Remark. The idea of the proof of Theorem 3 is similar to that of the proof of Theorem 1 in [2].

REFERENCES

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[3] K. Urbanik, *On algebraic operations in idempotent algebras*, ibidem 13 (1965), p. 129–157.

Reçu par la Rédaction le 24. 11. 1969