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ON THE DIMENSION OF REMAINDERS IN EXTENSIONS OF PRODUCT SPACES

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All spaces under discussion are metrizable.

1. Introduction. An extension Z of a space X is a space which contains X as a dense subset. The remainder of an extension Z of X is the space $Z\setminus X$.

In this paper* we discuss the lower bounds for the dimension of remainders of complete and compact extensions of products $X\times Y$.

The results can be summarized as follows. If X is not complete (not locally compact) and Y is σ-compact, then the dimension of Y is a lower bound for the dimension of the remainder of a complete extension (a compactification) of $X\times Y$. In case X is not complete, the following stronger result can be obtained. The remainder of a complete extension of $X\times Y$ contains uncountable many pairwise disjoint copies of Y. In case X is nowhere locally compact and of the second category no requirements have to be imposed on Y to show that the dimension of Y is a lower bound for the dimension of the remainder of a compact extension of $X\times Y$.

Throughout, $B_X(U)$ (or $B(U)$ when no confusion is likely to arise) denotes the boundary of U in X. The closure operator will be denoted by an upper bar. The strong inductive dimension of X is denoted by $\dim X$. *Complete* means *topologically complete.*

2. Complete extensions. We have

THEOREM 1. *Suppose X is not complete and Y is σ-compact. Suppose Z is a complete extension of $X\times Y$.*

Then $Z\setminus(X\times Y)$ contains uncountable many pairwise disjoint copies of Y.

COROLLARY 1. *Suppose X is not complete and Y is the union of a locally countable collection of compact sets. Suppose Z is a complete extension of $X\times Y$.*

Then $\dim Z\setminus(X\times Y)\geqslant \dim Y$.

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Before we state the next corollary we first mention a few definitions. A space is said to be *weakly countable-dimensional* if it is the union of a countable collection of closed finite-dimensional subsets. A space is called *countable-dimensional* if it is the union of a countable collection of zero-dimensional subsets. We say that a space is *weakly infinite-dimensional* if for every pair of countable closed collections $\{F_i \mid i = 1, 2, \ldots\}$ and $\{G_i \mid i = 1, 2, \ldots\}$ with $F_i \cap G_i = \emptyset$ $(i = 1, 2, \ldots)$ there exists an open collection $\{V_i \mid i = 1, 2, \ldots\}$ such that $F_i \subset V_i \subset X \setminus G_i$ and $\cap\{B(V_i) \mid i = 1, 2, \ldots\} = \emptyset$. If a space is not weakly infinite-dimensional, then it is strongly infinite-dimensional.

COROLLARY 2. *Suppose X is not complete and Y is compact. Suppose Z is a complete extension of $X \times Y$.*

If $Z \setminus (X \times Y)$ is 1. *weakly infinite-dimensional,* 2. *countable-dimensional, or* 3. *weakly countable-dimensional, then Y is* 1. *weakly infinite-dimensional,* 2. *countable-dimensional, or* 3. *weakly countable-dimensional.*

As a consequence of Corollary 2 we infer that the remainder in any compactification of the product of the rationals and the Hilbert cube is strongly infinite-dimensional. This answers a question posed by Lelek in [6]. As shown in example 2, the condition that X is not complete cannot be deleted.

We need the following lemma (for a proof see [5], p. 15):

LEMMA 1. *Suppose Y is σ-compact. Let p denote the natural projection of $X \times Y$ onto X. If F is an F_σ-subset of $X \times Y$, then $p(F)$ is an F_σ-subset of X.*

Proof of Theorem 1. Let $\tilde{X}$ and $\tilde{Y}$ be complete extensions of X and Y respectively. Assume $X \times Y$ is embedded in $\tilde{X} \times \tilde{Y}$ in the natural way. According to the extension theorem of Lavrentiev ([4], p. 335), the identity map of $X \times Y$ can be extended to a homeomorphism of a G_δ-subset of the complete extension Z onto a G_δ-subset G of $\tilde{X} \times \tilde{Y}$. It is sufficient to show that $G \setminus (X \times Y)$ contains uncountable many pairwise disjoint copies of Y. Let $F = (\tilde{X} \times \tilde{Y}) \setminus G$. F is an F_σ-subset of $\tilde{X} \times \tilde{Y}$. It follows that $F \cap (\tilde{X} \times Y)$ is an F_σ-subset of $\tilde{X} \times Y$. By the lemma above, $H = p\big(F \cap (\tilde{X} \times Y)\big)$, where p denotes the natural projection of $\tilde{X} \times Y$ onto $\tilde{X}$, is an F_σ-subset of $\tilde{X}$. Let $Q = \tilde{X} \setminus (X \cup H)$. Because $X = \tilde{X} \setminus (Q \cup H)$ and X is not complete, Q is uncountable. $Q \times Y \subset G \setminus (X \times Y)$ and the theorem follows.

Proof of Corollary 1. Let $Y = \bigcup\{C_a \mid a \in A\}$, where $\{C_a \mid a \in A\}$ is locally countable and each C_a is compact.

In case Y is finite-dimensional, by the sum theorem of dimension theory ([7], Theorem II. 1), for at least one C_a we have $\dim C_a = \dim Y$.

By Theorem 1, for any complete extension Z of $X \times Y$ the remainder contains a copy of C_a. It follows that $\dim Z \setminus (X \times Y) \geqslant \dim C_a = \dim Y$.

In case Y is infinite-dimensional, for each integer n there is a C_a with $\dim C_a \geqslant n$. Otherwise X is finite-dimensional by the sum theorem.

By Theorem 1 it follows that $\dim Z \setminus (X \times Y) \geqslant n$ for each integer n. Hence $Z \setminus (X \times Y)$ is infinite-dimensional.

Remark. Corollary 1 has already been proved in [1] section 2.4. In the proof a weaker version of Theorem 1 is used which, however, is not stated explicitely as a theorem.

Proof of Corollary 2. From Theorem 1 it follows that the remainder $Z \setminus (X \times Y)$ contains a copy of Y which is closed in the remainder since Y is compact. Because properties 1, 2 and 3 are invariant for the taking of closed subsets, the corollary follows.

3. Compactifications. In this section we discuss the lower bounds for the dimension of remainders of compactifications of products $X \times Y$. In discussing compactifications we may assume that all spaces are separable. Observe that every compactification is a complete extension. Thus Theorem 1 and Corollaries 1 and 2 hold also for compactifications.

Now, we relax the condition on X in Corollary 1 and obtain a similar result for compactifications:

THEOREM 2. *Suppose X is not locally compact and Y is the union of a locally countable collection of compact sets. Suppose Z is a compactification of $X \times Y$. Then* $\dim Z \setminus (X \times Y) \geqslant \dim Y$.

Theorem 2 is a generalization of [6], Theorem 1. However, the proof as outlined in [6] is not clear to the author. Here we present a totally different proof which also enables us to proof the following theorem:

THEOREM 3. *Suppose X is nowhere locally compact and of the second category. Suppose Z is a compactification of $X \times Y$.*

Then $\dim Z \setminus (X \times Y) \geqslant \dim Y$.

As shown by examples below the results of Theorems 2 and 3 cannot be strengthened to a result similar to that in Theorem 1.

For the proof of Theorems 2 and 3 we make use of the following lemma:

LEMMA 2. *If a subset X of a space Y has dimension $\leqslant n$, then for every pair of closed collections $\{F_i \mid i = 1, \ldots, n+1\}$ and $\{G_i \mid i = 1, \ldots, n+1\}$ of Y with $F_i \cap G_i = \emptyset$ there exists an open collection $\{V_i \mid i = 1, \ldots, n+1\}$ such that $F_i \subset V_i \subset X \setminus G_i$ and $[\bigcap \{B(V_i) \mid i = 1, \ldots, n+1\}] \cap X = \emptyset$.*

Moreover, if X is closed, then the converse holds.

Proof. If $X = Y$, the lemma is a version of [7], corollary of theorem II. 8. It is obvious that the converse holds for closed subsets X of Y.

Suppose $\dim X \leqslant n$. Let $\{F_i \mid i = 1, \ldots, n+1\}$ and $\{G_i \mid i = 1, \ldots,$

$\ldots, n+1\}$ be closed collections with $F_i \cap G_i = \emptyset$. For each i open sets U_i and W_i are selected such that $F_i \subset U_i$, $G_i \subset W_i$ and $\bar{U}_i \cap \bar{W}_i = \emptyset$. Because $\dim X \leqslant n$, in the subspace X there exists an open collection $\{D_i | i = 1, \ldots, n+1\}$ such that $\bar{U}_i \cap X \subset D_i \subset X \setminus \bar{W}_i$ and $\cap \{B_X(D_i) | i = 1, \ldots, n+1\} = \emptyset$. For each i neither of the sets $F_i \cup D_i$ and $G_i \cup \cup (X \setminus \bar{D}_i)$ contains a cluster point of the other. By the heriditary normality of Y (cf. [7], p. 3) there exist open sets V_i such that $F_i \cup D_i \subset V_i$ and $\bar{V}_i \cap (G_i \cup (X \setminus \bar{D})) = \emptyset$. $B_Y(V_i) = \bar{V}_i \setminus V_i$ and $B_Y(V_i) \cap X \subset B_X(D)$. The lemma follows.

Proof of Theorems 2 and 3. Suppose $\dim Y \geqslant n$ and $\dim Z \setminus (X \times Y) < n$. We shall derive a contradiction. By the lemma, in Y there exist closed collections $\{F_i | i = 1, \ldots, n\}$ and $\{G_i | i = 1, \ldots, n\}$ with $F_i \cap G_i = \emptyset$ such that for each open collection $\{V_i | i = 1, \ldots, n\}$ with $F_i \subset V_i \subset X \setminus G_i$ we have $\bigcap \{B_Y(V_i) | i = 1, \ldots, n\} \neq \emptyset$. Let $L_i = X \times F_i$ and $K_i = X \times \times G_i$, $i = 1, \ldots, n$. Let $A = \bigcup \{\bar{L}_i \cap \bar{K}_i | i = 1, \ldots, n\}$, where the upper bar denotes the closure in Z. A is a closed subset of Z and $Z \setminus A$ is locally compact. Moreover, in $Z \setminus A$ the sets L_i and K_i have disjoint closures which will be denoted by L_i' and K_i' respectively, $i = 1, \ldots, n$. Because $\dim Z \setminus (A \cup X \times Y) < n$, in the space $Z \setminus A$ there exists an open collection $\{V_i | i = 1, \ldots, n\}$ such that $L_i' \subset V_i \subset (Z \setminus A) \setminus K_i'$ and $[\bigcap \{B_{Z \setminus A}(V_i) | i = 1, \ldots, n\}] \cap [Z \setminus (A \cup X \times Y)] = \emptyset$. It follows that $D = \bigcap \{B_{Z \setminus A}(V_i) | i = 1, \ldots, n\} \subset X \times Y$. Let π_X and π_Y denote the natural projections of $X \times Y$ onto X and Y respectively. Let $\pi = \pi_X | D$. Clearly π is continuous. First, we show that π maps D *onto* X. For every point $x \in X$ the map π_Y restricted to $\pi_X^{-1}(x) = Y_x$ is a homeomorphism between Y_x and Y. Observe that $\pi_Y(L_i \cap Y_x) = F_i$, $\pi_Y(K_i \cap Y_x) = G_i$ and $\{\pi_Y(V_i \cap Y_x) | i = 1, \ldots, n\}$ is an open collection in Y with $F_i \subset \pi_Y(V_i \cap Y_x) \subset Y \setminus G_i$.

It follows that $\bigcap \{B_Y(\pi_Y(V_i \cap Y_x)) | i = 1, \ldots, n\} \neq \emptyset$ and $D \cap Y_x \neq \emptyset$. This shows that π maps D onto X.

Now in order to prove Theorem 2 we proceed as follows. We first assume that Y is compact. Then π is also closed, because π_X is closed and π is the restriction of π_X to a closed subset. Moreover, the inverse image of each point under π is compact. D is a closed subset of the locally compact set $Z \setminus A$. It follows that $\pi(D) = X$ is locally compact which is a contradiction. From this result Theorem 2 can be deduced in the same way as Corollary 1 is obtained from Theorem 1.

Theorem 3 is proved as follows. D is locally compact and therefore σ-compact. Because π is continuous, $X = \pi(D)$ is σ-compact. However, this contradicts the fact that X is nowhere locally compact and of the second category.

Example 1. By means of examples it is shown that the results of Theorems 2 and 3 cannot be strengthened to a result similar to that in Theorem 1.

For each integer $k \geqslant 1$, let A_k be the topological union of k intervals of length $1/2^k$. B_k is obtained by identifying k endpoints—one out of each interval—to one point p_k. The space Y is obtained as follows. Take the topological union of the unit interval $[0,1]$ and the B_k, $k = 1, 2, \ldots$ The point p_k is identified with $1/k \in [0,1]$. Let π denote the identification map. The resulting space Y is compact and for each $k \geqslant 2$ has exactly one point of order $k+2$.

Now let X be the closed unit interval $[0,1]$ with the points $1/n$, $n = 1, 2, \ldots$, deleted.

The space $X \times Y$ can be compactified by $[0,1] \times Y$. Let Z be the quotient space of $[0,1] \times Y$ which is obtained by identifying each of the sets

$$\left\{\frac{1}{n}\right\} \times \left[\pi\left[0, \frac{1}{n}\right] \cup [\bigcup \{\pi(B_k) \mid k \geqslant n\}]\right]$$

to a point c_n, $n = 1, 2, \ldots$; Z is a compactification of $X \times Y$ and $Z \setminus (X \times Y)$ contains no copy of Y as is easily seen. X and Y satisfy the conditions of Theorem 2.

A small modification of this example shows that Theorem 3 cannot be strengthened. Let X be the closed unit interval $[0,1]$ with the rational points p/q deleted. We assume p and q have no common divisors except 1. Let Z be the quotient space of $[0,1] \times Y$ which is obtained by identifying each of the sets

$$\left\{\frac{p}{q}\right\} \times \left[\pi\left[0, \frac{1}{q}\right] \cup [\bigcup \{\pi(B_k) \mid k \geqslant q\}]\right]$$

to a point $c_{p,q}$. (It is assumed that 0 is expressed as $0/1$.) Then Z is a compactification of $X \times Y$ the remainder of which contains no copy of Y. X satisfies the conditions of Theorem 3.

Example 2. Here we mention an example which shows that in Corollary 2 the condition that X is not complete cannot be deleted. This example has been introduced for other purposes by Anderson in [3], lemma 5.2.

Let $\mathfrak{l}_2$ be the separable Hilbert space. $\mathfrak{l}_2 = \{(x_i)_{i>0} \mid x_i$ is a real number and $\Sigma x_i^2 < \infty\}$. Consider the infinite-dimensional ellipsoid $E = \{(x_i) \in \mathfrak{l}_2 \mid \Sigma i^2 x_i^2 \leqslant 1\}$ and let $B = \{(x_i) \in \mathfrak{l}_2 \mid \Sigma i^2 x_i^2 = 1$ and only finitely many x_i are non-zero$\}$.

E is homeomorphic to the Hilbert cube I^∞ and $E \setminus B$ is homeomorphic to the countable infinite product of real lines $\mathfrak{s}$. Observe that B is weakly infinite-dimensional. Since $\mathfrak{s}$ is known to be homeomorphic to $\mathfrak{s} \times I^\infty$ (see [2]), this shows that the product of $\mathfrak{s}$ and a compact strongly infinite-dimensional space has a compactification with a weakly infinite-dimensional remainder.

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