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*ON DIFFERENTIABILITY OF PEANO TYPE FUNCTIONS**

BY

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In this paper we investigate the properties of Peano functions (on the real line R), i.e. vector functions $F = (f_1, f_2)\colon R \to R^2$ such that $F(R) = R^2$.

Let $A \subseteq R$, $f\colon A \to R$ and $E \subseteq A$. We say that f fulfils the *Banach condition* (T_2) on E $(f \in T_2(E))$ if

$$\lambda(\{y \in f(E)\colon |f^{-1}(\{y\}) \cap E| > \aleph_0\}) = 0,$$

where λ denotes the Lebesgue measure. We also write $f \in VB(E)$ if f is of bounded variation on E, and $f \in VBG(E)$ if E is the sum of a countable sequence of sets E_n, where $f \in VB(E_n)$ for each n ([2], Chap. VII, p. 221).

Let M_1, M_2 be any sets and let $S \subseteq M_1 \times M_2$. Assume that $u \in M_1$ and $v \in M_2$. We put

$$S_u = \{y \in M_2\colon (u, y) \in S\} \quad \text{and} \quad S^v = \{x \in M_1\colon (x, v) \in S\}.$$

Throughout this paper we consider only finite derivatives of functions.

THEOREM 1. *The existence of a Peano function $F = (f_1, f_2)$ such that for each $x \in R$ there exists at least one of the derivatives $f_1'(x)$, $f_2'(x)$ is equivalent to the Continuum Hypothesis.*

Proof. We use the following theorem of Sierpiński:

Let M_1, M_2 be sets of power $\mathfrak{c}$. The existence of sets $S_1, S_2 \subseteq M_1 \times M_2$ such that $S_1 \cup S_2 = M_1 \times M_2$ and that the sets $(S_1)_u$, $(S_2)^v$ are countable for each $u \in M_1$ and $v \in M_2$ is equivalent to the Continuum Hypothesis (CH) ([3], Chap. I, Proposition P_1, p. 9).

Assume CH and take sets S_1, S_2 of Sierpiński's theorem applied to $M_1 = M_2 = R$.

* Ce travail fut mal composé dans le fascicule 48.2. Il fut reproduit sur une feuille non numérotée et insérée dans le fascicule 49.1. A présent nous le reproduisons encore une fois en l'intégrant entièrement au corps de ce fascicule. *La Rédaction*

Let $\varphi(x) = x \sin x$ for $x \in R$. For $u, v \in R$ let

$$\varphi^{-1}(\{u\}) \cap (-\infty, -1\rangle = \{t_1^u, t_2^u, \ldots\}, \quad \varphi^{-1}(\{v\}) \cap \langle 1, \infty) = \{s_1^v, s_2^v, \ldots\},$$
$$(S_1)_u = \{y_1^u, y_2^u, \ldots\}, \quad (S_2)^v = \{x_1^v, x_2^v, \ldots\}.$$

For $t \in (-\infty, 1)$ we put $f_1(t) = \varphi(t)$. If $t \in \langle 1, \infty)$, then $t = s_n^v$ for some real v and natural n. Let us put $f_1(s_n^v) = x_n^v$.

Similarly, for $t \in (-1, \infty)$ we put $f_2(t) = \varphi(t)$ and $f_2(t_n^u) = y_n^u$ for $t_n^u \in \langle -\infty, -1\rangle$.

One can check that the function $F = (f_1, f_2)$ is the one looked for.

Conversely, assume now that $F(R) = R^2$ and $D_1 \cup D_2 = R$, where $D_i = \{t \in R: f_i'(t) \text{ exists}\}$ $(i = 1, 2)$. The functions f_1, f_2 satisfy the Banach condition (T_2) on the sets D_1, D_2, respectively (see [2], Chap. VII, Theorem 10.1, p. 234, and Chap. IX, p. 279([1])). Hence the sets

$$N_i = \{y \in f_i(D_i): |f_i^{-1}(\{y\}) \cap D_i| > \aleph_0\}, \quad i = 1, 2,$$

have Lebesgue measure zero. Therefore, the sets $M_i = R - N_i$ $(i = 1, 2)$ are of power $\mathfrak{c}$. Let $S_i = F(D_i) \cap (M_1 \times M_2)$, $i = 1, 2$. The sets S_1, S_2, M_1, M_2 satisfy the conditions of Sierpiński's theorem, and hence the proof of the theorem is complete.

We shall see further that a function $F = (f_1, f_2)$ defined in Theorem 1 does not exist when we assume that at least one of the coordinate functions f_1 or f_2 is Lebesgue measurable. This will follow from Theorem 3. First we prove the following

THEOREM 2. *Let $F = (f_1, f_2)$, where $f_1 \in T_2(R)$ and f_2 is an arbitrary function. Let $F(R)$ be a Lebesgue measurable subset of R^2. Then $\lambda_2(F(R)) = 0$, where λ_2 is the Lebesgue measure on the plane R^2.*

Proof. There exist two disjoint sets A and B such that $A \cup B = R$, $\lambda(B) = 0$, and $|f_1^{-1}(\{y\})| \leqslant \aleph_0$ for each $y \in A$. According to Fubini's theorem we can write

$$\begin{aligned}\lambda_2(F(R)) &= \lambda_2(F(R) \cap (A \times R)) \\ &= \int_A \lambda(F(R) \cap (\{x\} \times R))\, d\lambda(x) \\ &= \int_A \lambda(f_2(f_1^{-1}(\{x\})))\, d\lambda(x) = 0\end{aligned}$$

because $|f_2(f_1^{-1}(\{x\}))| \leqslant \aleph_0$ for each $x \in A$.

COROLLARY. *Let $f_1 \in VBG(R)$ and let f_2 be continuous on R. Assume that $F = (f_1, f_2)$. Then $\lambda_2(F(R)) = 0$.*

([1]) In [2] this fact is shown for intervals, but it is true for any subset of R.

Proof. Let $\{E_n: n = 1, 2, \ldots\}$ be a family of sets such that

$$\bigcup_{n=1}^{\infty} E_n = R \quad \text{and} \quad f_1 \in VB(E_n) \text{ for each } n.$$

Let us consider any fixed set E_n. The function $f \upharpoonright E_n$ can be extended to a function $g_n \in VB(R)$ ([2], Chap. VII, Lemma 4.1, p. 221). Let $F_n = (g_n, f_2)$. Since F_n is a Borel function, the set $F_n(R)$ is analytic ([1], Chap. III, Section 38, Proposition 5, p. 457). Therefore, the set $F_n(R)$ is Lebesgue measurable ([1], Chap. III, Section 39, p. 482). Hence, by Theorem 2, we have $\lambda_2(F_n(R)) = 0$, which implies $\lambda_2(F_n(E_n)) = \lambda_2(F(E_n)) = 0$. Finally, $\lambda_2(F(R)) = 0$.

THEOREM 3. *Let f_1: $R \to R$ and f_2: $R \to R$. Assume that*

(i) *the function f_1 is Lebesgue measurable;*

(ii) *for each $x \in R$ there exists at least one of the derivatives $f_1'(x)$, $f_2'(x)$;*

(iii) *$F(R)$ is a Lebesgue measurable subset of R^2, where $F = (f_1, f_2)$.*

Then $\lambda_2(F(R)) = 0$.

Proof. Let us put $D_i = \{t \in R: f_i'(t) \text{ exists}\}$, $i = 1, 2$. There exists a sequence $\{K_n\}_{n=1}^{\infty}$ of closed subsets of R such that $\lambda(R - K_n) < 1/n$ and $f_1 \upharpoonright K_n$ is continuous for $n = 1, 2, \ldots$ Let us consider the set $D_2 \cap K_n$ for a certain fixed n. The function f_2 is differentiable on $D_2 \cap K_n$, and so $f_2 \in VBG(D_2 \cap K_n)$ ([2], Chap. VII, Theorem 10.1, p. 234). Let $\{A_j: j = 1, 2, \ldots\}$ be a family of sets such that

$$D_2 \cap K_n = \bigcup_{j=1}^{\infty} A_j \quad \text{and} \quad f_2 \in VB(A_j) \quad \text{for } j = 1, 2, \ldots$$

For every j there exists an extension of $f_2 \upharpoonright A_j$ to a function $g_j \in VB(R)$. Of course, there also exists an extension of $f_1 \upharpoonright K_n$ to a continuous function h on R. For the vector function $H = (h, g_j)$ we have $\lambda_2(H(R)) = 0$ (see the Corollary), whence $\lambda_2(H(A_j)) = 0$. This implies $\lambda_2(F(D_2 \cap K_n)) = 0$ and, consequently

$$\lambda_2\Big(F\big(D_2 \cap \bigcup_{n=1}^{\infty} K_n\big)\Big) = 0.$$

The function f_2 is differentiable on the set $D_2 - \bigcup_{n=1}^{\infty} K_n$ and $\lambda(D_2 - \bigcup_{n=1}^{\infty} K_n) = 0$, whence

$$\lambda\Big(f_2\big(D_2 - \bigcup_{n=1}^{\infty} K_n\big)\Big) = 0$$

([2], Chap. VII, Theorem 6.5, p. 227). Consequently, we obtain

$$\lambda_2\Big(F\big(D_2 - \bigcup_{n=1}^{\infty} K_n\big)\Big) = 0$$

and, finally, $\lambda_2(F(D_2)) = 0$. Thus $F(D_1)$ is Lebesgue measurable. Let us put $\varphi(t) = f_1(t)$ for $t \in D_1$ and $\varphi(t) = 0$ for $t \in R - D_1$. The function φ satisfies the Banach condition (T_2) on R. Let $G = (\varphi, f_2)$. The sets $G(D_1) = F(D_1)$ and $G(R - D_1)$ are Lebesgue measurable, and from Theorem 2 we obtain $\lambda_2(G(R)) = 0$. Hence $\lambda_2(F(D_1)) = 0$. Finally, $\lambda_2(F(R)) = 0$.

We formulate now other versions of Theorems 2 and 3, omitting the assumption of Lebesgue measurability of $F(R)$. To prove these theorems we should apply the same methods as those used in the proofs of Theorems 2 and 3.

THEOREM 2'. *Let $F = (f_1, f_2)$, where $f_1 \in T_2(R)$ and f_2 is an arbitrary function. Then $\lambda_2^i(F(R)) = 0$, where λ_2^i denotes the inner Lebesgue measure on the plane R^2.*

THEOREM 3'. *Let $f_1\colon R \to R$, $f_2\colon R \to R$, and $F = (f_1, f_2)$. Assume that the function f_1 is Lebesgue measurable and that for each $x \in R$ there exists at least one of the derivatives $f_1'(x)$, $f_2'(x)$. Then $\lambda_2^i(F(R)) = 0$.*

Finally, we pose the following problem:

PROBLEM (**P 1276**). Does there exist a function $F = (f_1, f_2)\colon I \to I \times I$, where $I = \langle 0, 1 \rangle$, such that $F(I) = I \times I$ and for each $x \in I$ there exists at least one of the derivatives $f_1'(x)$, $f_2'(x)$ (as in Theorem 1 one can prove that from the existence of F CH would follow)?

Let us mention that if we put above an open or half-open interval (instead of I) as the domain of F, then the existence of F is, as in Theorem 1, equivalent to CH.

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