COLLOQUIUM MATHEMATICUM

VOL. XXXIV 1976 FASC. 2

ON FIXED-POINT THEOREMS IN COMPLETE METRIC SPACES

BY

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1. Introduction. The well-known Banach's Fixed-Point Theorem [2] states that each contraction mapping f of a complete metric space (X, d) into itself has a unique fixed point. This result has been generalized in many ways by various authors, see Belluce and Kirk [1], Dass and Gupta [3], Edelstein [4], Sehgal [6], Guseman [5], and others. Sehgal investigated mappings having a contractive iterate at each point of the space and proved the following theorem:

Let (X, d) be a complete metric space and let $f\colon X \to X$ be a continuous mapping satisfying the following condition: there exists a $k < 1$ such that, for each $x \in X$, there is an integer $n(x) \geqslant 1$ such that, for all $y \in X$,

$$d\big(f^{n(x)}(x), f^{n(x)}(y)\big) \leqslant kd(x, y). \tag{1}$$

Then f has a unique fixed point u and $f^n(x_0) \to u$ for each $x_0 \in X$.

Guseman [5] generalized this result for mappings which are not necessarily continuous but satisfy (1) on a subset of the space.

In this paper* we have taken a somewhat different type of condition instead of (1) and have shown the existence of the unique fixed point following the lines of arguments of Sehgal [6]. In the last section we generalize our result to mappings which are not necessarily continuous but satisfy a rather weaker condition.

2. We prove the following

THEOREM 1. *Let f be a continuous mapping of a complete metric space (X, d) into itself satisfying the following condition: for each $x \in X$, there is an integer $n(x) \geqslant 1$ such that, for all $y \in X$,*

$$d\big(f^{n(x)}(x), f^{n(x)}(y)\big) \leqslant ad\big(x, f^{n(x)}(y)\big) + \beta d\big(y, f^{n(x)}(x)\big), \tag{2}$$

where $a > 0$, $\beta > 0$ and $a+\beta < \frac{1}{2}$.

Then f has a unique fixed point u and $f^n(x_0) \to u$ for each $x_0 \in X$.

* The authors are thankful to C.S.I.R. (India) for providing research fellowships.

In order to prove this theorem, we first prove the following

LEMMA. *If $f\colon X \to X$ is a mapping satisfying the conditions of Theorem 1, then*

$$r(x) = \sup_n d\big(f^n(x), x\big) \tag{3}$$

is finite for each $x \in X$.

Proof. Let $x \in X$ and let

$$\delta(x) = \max\big\{d\big(f^{\varrho}(x), x\big)\colon\ \varrho = 1, 2, \ldots, n(x)\big\}.$$

If n is a positive integer, then there exists an integer $t \geqslant 0$ such that $tn(x) < n \leqslant (t+1)n(x)$ for $n > n(x)$ and

$$\begin{aligned} d\big(f^n(x), x\big) &\leqslant d\big(f^{n(x)}f^{n-n(x)}(x), f^{n(x)}(x)\big) + d\big(f^{n(x)}(x), x\big) \\ &\leqslant \alpha d\big(f^{n-n(x)}(x), f^{n(x)}(x)\big) + \beta d\big(x, f^n(x)\big) + d\big(f^{n(x)}(x), x\big), \end{aligned}$$

i.e.,

$$\begin{aligned} d\big(f^n(x), x\big) &\leqslant \frac{\alpha}{1-\beta} d\big(f^{n-n(x)}(x), x\big) + \frac{1+\alpha}{1-\beta}\delta(x) \\ &\leqslant \left[\frac{1+\alpha}{1-\beta} + \frac{\alpha(1+\alpha)}{(1-\beta)^2} + \ldots + \frac{\alpha^t(1+\alpha)}{(1-\beta)^{t+1}}\right]\delta(x) \\ &< \frac{1+\alpha}{1-\beta}\left[1 + \frac{\alpha}{1-\beta} + \left(\frac{\alpha}{1-\beta}\right)^2 + \ldots\right]\delta(x) \\ &= \frac{1+\alpha}{1-\beta}\,\frac{1}{1-\alpha/(1-\beta)}\,\delta(x) = \frac{1+\alpha}{1-(\alpha+\beta)}\,\delta(x) \end{aligned}$$

for all $n \geqslant 0$.

Hence (3) is finite.

Proof of Theorem 1. Let $x_0 \in X$ be arbitrary. Let $m_0 = n(x_0)$, $x_1 = f^{m_0}(x_0)$ and, successively, $m_i = n(x_i)$, $x_{i+1} = f^{m_i}(x_i)$. We show that $\{x_n\}$ is a convergent sequence. We have

$$\begin{aligned} d(x_1, x_2) &= d\big(f^{m_0}(x_0), f^{m_0}f^{m_1}(x_0)\big) \\ &\leqslant \alpha d(x_0, x_1) + \alpha d(x_1, x_2) + \beta d\big(f^{m_1}(x_0), x_0\big) + \beta d(x_0, x_1), \end{aligned}$$

i.e.,

$$d(x_1, x_2) \leqslant a d(x_0, x_1) + b d\big(f^{m_1}(x_0), x_0\big), \quad \text{where } a = \frac{\alpha+\beta}{1-\alpha},\ b = \frac{\beta}{1-\alpha}.$$

Again,

$$\begin{aligned} d(x_2, x_3) &= d\big(f^{m_1}(x_1), f^{m_1}f^{m_2}(x_1)\big) \\ &\leqslant \alpha d(x_1, x_2) + \alpha d(x_2, x_3) + \beta d\big(x_1, f^{m_2}(x_1)\big) + \beta d(x_1, x_2), \end{aligned}$$

i.e.,

$$d(x_2, x_3) \leqslant a d(x_1, x_2) + b d\big(x_1, f^{m_2}(x_1)\big).$$

However,

$$\begin{aligned} d\big(x_1, f^{m_2}(x_1)\big) &= d\big(f^{m_0}(x_0), f^{m_0}f^{m_2}(x_0)\big) \\ &\leqslant ad(x_0, x_1) + ad\big(x_1, f^{m_2}(x_1)\big) + \beta d\big(f^{m_2}(x_0), x_0\big) + \beta d(x_0, x_1), \end{aligned}$$

i.e.,

$$d\big(x_1, f^{m_2}(x_1)\big) \leqslant ad(x_0, x_1) + bd\big(f^{m_2}(x_0), x_0\big),$$

so that

$$d(x_2, x_3) \leqslant a(a+b)d(x_0, x_1) + abd\big(f^{m_1}(x_0), x_0\big) + b^2 d\big(f^{m_2}(x_0), x_0\big).$$

Similarly,

$$\begin{aligned} d(x_3, x_4) \leqslant a(a+b)^2 d(x_0, x_1) + ab(a+b)d\big(f^{m_1}(x_0), x_0\big) + \\ + ab^2 d\big(f^{m_2}(x_0), x_0\big) + b^3 d\big(f^{m_3}(x_0), x_0\big) \end{aligned}$$

and, in general,

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant a(a+b)^{n-1} d(x_0, x_1) + ab(a+b)^{n-2} d\big(f^{m_1}(x_0), x_0\big) + \\ &\quad + ab^2(a+b)^{n-3} d\big(f^{m_2}(x_0), x_0\big) + \ldots + \\ &\quad + ab^{n-1} d\big(f^{m_{n-1}}(x_0), x_0\big) + b^n d\big(f^{m_n}(x_0), x_0\big) \\ &\leqslant [a(a+b)^{n-1} + ab(a+b)^{n-2} + \ldots + ab^{n-1} + b^n] r(x_0). \end{aligned}$$

Now, for $k > n$,

$$\begin{aligned} d(x_n, x_k) &\leqslant d(x_n, x_{n+1}) + \ldots + d(x_{k-1}, x_k) \\ &\leqslant a[(a+b)^{n-1} + b(a+b)^{n-2} + \ldots + b^{n-2}(a+b) + b^{n-1}] \times \\ &\quad \times [1 + (a+b) + (a+b)^2 + \ldots + (a+b)^{k-n-1}] r(x_0) + \\ &\quad + ab^n [1 + (a+b) + (a+b)^2 + \ldots + (a+b)^{k-n-2}] r(x_0) + \\ &\quad + ab^{n+1} [1 + (a+b) + (a+b)^2 + \ldots + (a+b)^{k-n-3}] r(x_0) + \\ &\quad + \ldots + ab^{k-3} [1 + (a+b)] r(x_0) + ab^{k-2} r(x_0) + \\ &\quad + b^n [1 + b + b^2 + \ldots + b^{k-n-1}] r(x_0) \\ &< \Big[a \sum_{i=0}^{n-1} b^i (a+b)^{n-i-1}\Big]\Big[\frac{1}{1-(a+b)}\Big] r(x_0) + \\ &\quad + \frac{ab^n}{1-(a+b)} r(x_0) + \frac{ab^{n+1}}{1-(a+b)} r(x_0) + \ldots + \\ &\quad + \frac{ab^{k-3}}{1-(a+b)} r(x_0) + ab^{k-2} r(x_0) + \frac{b^n}{1-b} r(x_0) \to 0 \\ &\qquad\qquad \text{as } n, k \to \infty, \end{aligned}$$

showing thereby that $\{x_n\}$ is a Cauchy sequence.

Let $x_n \to u \in X$. We show that $f(u) = u$. If $f(u) \neq u$, then there exists a pair of disjoint closed neighbourhoods M and N such that $u \in M, f(u) \in N$, and

$$\delta = \inf\{d(x, y): x \in M, y \in N\} > 0. \tag{4}$$

Since f is continuous, $x_n \in M$ and $f(x_n) \in N$ for all n sufficiently large. Now,

$$\begin{aligned} d\big(f(x_n), x_n\big) &= d\big(f^{m_n-1}f(x_{n-1}), f^{m_n-1}(x_{n-1})\big) \\ &\leqslant ad\big(f(x_{n-1}), x_n\big) + \beta d\big(x_{n-1}, f(x_n)\big) \\ &\leqslant ad\big(f(x_{n-1}), x_{n-1}\big) + ad(x_{n-1}, x_n) + \beta d(x_{n-1}, x_n) + \\ &\quad + \beta d\big(x_n, f(x_n)\big), \end{aligned}$$

i.e.,

$$\begin{aligned} d\big(f(x_n), x_n\big) &\leqslant \frac{a}{1-\beta} d\big(f(x_{n-1}), x_n\big) + \frac{a+\beta}{1-\beta} d(x_{n-1}, x_n) \leqslant \ldots \\ &\leqslant \left(\frac{a}{1-\beta}\right)^n d\big(f(x_0), x_0\big) + \left(\frac{a}{1-\beta}\right)^{n-1} \frac{a+\beta}{1-\beta} d(x_0, x_1) + \\ &\quad + \left(\frac{a}{1-\beta}\right)^{n-2} \frac{a+\beta}{1-\beta} d(x_1, x_2) + \ldots + \\ &\quad + \frac{a}{1-\beta}\,\frac{a+\beta}{1-\beta} d(x_{n-2}, x_{n-1}) + \frac{a+\beta}{1-\beta} d(x_{n-1}, x_n) \to 0 \\ &\qquad \text{as } n \to \infty, \end{aligned}$$

contradicting (4). Hence $f(u) = u$.

To prove uniqueness, assume that there exists $v \in X$, $v \neq u$, such that $f(v) = v$. Then

$$\begin{aligned} d(u, v) &= d\big(f^{n(u)}(u), f^{n(u)}(v)\big) \\ &\leqslant ad(u, v) + ad\big(v, f^{n(u)}(v)\big) + \beta d(v, u) + \beta d\big(u, f^{n(u)}(u)\big) \\ &= (a+\beta) d(u, v). \end{aligned}$$

Hence $u = v$.

Finally, we show that $f^n(x_0) \to u$. Let

$$\delta^* = \max\big\{d\big(f^s(x_0), u\big): s = 0, 1, 2, \ldots, \big(n(u)-1\big)\big\}.$$

If n is a sufficiently large integer, then $n = rn(u) + q$, $0 \leqslant q < n(u)$, $r > 0$, and

$$\begin{aligned} d\big(f^n(x_0), u\big) &= d\big(f^{rn(u)+q}(x_0), f^{n(u)}(u)\big) \\ &\leqslant ad\big(f^{(r-1)n(u)+q}(x_0), u\big) + \beta d\big(u, f^n(x_0)\big), \end{aligned}$$

i.e.,

$$d\big(f^{n}(x_0),\,u\big)\leqslant\frac{a}{1-\beta}\,d\big(f^{(r-1)n(u)+q}(x_0),\,u\big)\leqslant\ldots$$
$$\leqslant\left(\frac{a}{1-\beta}\right)^{r}d\big(f^{q}(x_0),\,u\big)\leqslant\left(\frac{a}{1-\beta}\right)^{r}\delta^{*}.$$

Since $n\to\infty$ implies $r\to\infty$, we have

$$d\big(f^{n}(x_0),\,u\big)\to 0\quad\text{ as } n\to\infty.$$

This establishes the theorem completely.

3. In this section, we generalize Theorem 1 for mappings which are not continuous but satisfy (2) on a subset of the space.

THEOREM 2. *Let f be a mapping of a complete metric space (X,d) into itself. Suppose there exists $B\subset X$ such that*

(i) $f(B)\subset B$,

(ii) *for each $y\in B$, there is an integer $n(y)\geqslant 1$ with*

$$d\big(f^{n(y)}(x),f^{n(y)}(y)\big)\leqslant a d\big(x,f^{n(y)}(y)\big)+\beta d\big(y,f^{n(y)}(x)\big)\quad \textit{for all } x\in B,\ a>0,\ \beta>0,\ a+\beta<\tfrac{1}{2},$$

(iii) *for some $x_0\in B$, $\mathrm{Cl}\{f^{n}(x_0)\colon n\geqslant 1\}\subset B$, where* Cl *denotes the closure.*

Then there is a unique $u\in B$ such that $f(u)=u$ and $f^{n}(y_0)\to u$ for each $y_0\in B$. Furthermore, if

$$d\big(f^{n(u)}(x),f^{n(u)}(u)\big)\leqslant a d\big(x,f^{n(u)}(u)\big)+\beta d\big(u,f^{n(u)}(x)\big)\quad \textit{for all } x\in X,$$

then u is unique in X and $f^{n}(x_0)\to u$ for each $x_0\in X$.

In order to prove this theorem we first give the following

Remark. Let f be a mapping of a metric space (X,d) into itself. Let $B\subset X$ with $f(B)\subset B$. Suppose there exists a $u\in B$ and a positive integer $n(u)$ with

$$d\big(f^{n(u)}(x),f^{n(u)}(u)\big)\leqslant a d\big(x,f^{n(u)}(u)\big)+\beta d\big(u,f^{n(u)}(x)\big)\quad \text{for all } x\in B,\ a>0,\ \beta>0,\ a+\beta<\tfrac{1}{2}. \tag{5}$$

Further, if $f^{n}(u)=u$, then u is the unique fixed point of f in B, and $f^{n}(y_0)\to u$ for each $y_0\in B$, for (5) reduces to

$$d\big(f^{n(u)}(x),f^{n(u)}(u)\big)\leqslant a d(x,u)+\beta d\big(f^{n(u)}(u),f^{n(u)}(x)\big),$$

i.e.,

$$d\big(f^{n(u)}(x),f^{n(u)}(u)\big)\leqslant\frac{a}{1-\beta}d(x,u).$$

It now follows from the lemma proved by Guseman [5] that there is a unique $u \in B$ satisfying $f(u) = u$ and $f^n(y_0) \to u$ for each $y_0 \in B$.

Proof of Theorem 2. Let $y \in B$. By the Lemma of Section 2, it follows easily that

$$r(y) = \sup_n d\big(f^n(y), y\big)$$

is finite. For $x_0 \in B$ as taken in (iii), let $m_0 = n(x_0)$, $x_1 = f^{m_0}(x_0)$ and, successively, $m_i = n(x_i)$, $x_{i+1} = f^{m_i}(x_i)$.

As in Section 2, we have, by usual calculations,

$$d(x_n, x_{n+1}) \leqslant [a(a+b)^{n-1} + ab(a+b)^{n-2} + \ldots + ab^{n-1} + b^n]\, r(x_0)$$

and, for $k > n$,

$$d(x_n, x_k) \to 0 \quad \text{as } n, k \to \infty,$$

showing thereby that $\{x_n\}$ is a Cauchy sequence. Using completeness of X and (iii), we have $x_n \to u \in B$. Thus there is an integer $n(u) \geqslant 1$ such that

$$d\big(f^{n(u)}(y), f^{n(u)}(u)\big) \leqslant a d\big(y, f^{n(u)}(u)\big) + \beta d\big(u, f^{n(u)}(y)\big) \quad \text{for each } y \in B.$$

Now,

$$\begin{aligned} d\big(u, f^{n(u)}(u)\big) &\leqslant d(u, x_n) + d\big(x_n, f^{n(u)}(x_n)\big) + d\big(f^{n(u)}(x_n), f^{n(u)}(u)\big) \\ &\leqslant d(u, x_n) + d\big(x_n, f^{n(u)}(x_n)\big) + a d\big(x_n, f^{n(u)}(u)\big) + \beta d\big(u, f^{n(u)}(x_n)\big) \\ &\leqslant d(u, x_n) + d\big(x_n, f^{n(u)}(x_n)\big) + a d(u, x_n) + \\ &\qquad + a d\big(u, f^{n(u)}(u)\big) + \beta d(u, x_n) + \beta d\big(x_n, f^{n(u)}(x_n)\big), \end{aligned}$$

i.e.,

$$d\big(u, f^{n(u)}(u)\big) \leqslant \frac{1 + (a+\beta)}{1-a} d(u, x_n) + \frac{1+\beta}{1-a} d\big(x_n, f^{n(u)}(x_n)\big).$$

However,

$$\begin{aligned} d\big(x_n, f^{n(u)}(x_n)\big) &= d\big(f^{m_{n-1}}(x_{n-1}), f^{m_{n-1}} f^{n(u)}(x_{n-1})\big) \\ &\leqslant a d\big(x_{n-1}, f^{n(u)}(x_n)\big) + \beta d\big(f^{n(u)}(x_{n-1}), x_n\big) \\ &\leqslant a d(x_{n-1}, x_n) + a d\big(x_n, f^{n(u)}(x_n)\big) + \\ &\qquad + \beta d\big(x_{n-1}, f^{n(u)}(x_{n-1})\big) + \beta d(x_{n-1}, x_n), \end{aligned}$$

i.e.,

$$\begin{aligned} d\big(x_n, f^{n(u)}(x_n)\big) &\leqslant a d(x_{n-1}, x_n) + b d\big(x_{n-1}, f^{n(u)}(x_{n-1})\big) \leqslant \ldots \\ &\leqslant a d(x_{n-1}, x_n) + ab d(x_{n-2}, x_{n-1}) + ab^2 d(x_{n-3}, x_{n-2}) + \ldots + \\ &\qquad + b^n d\big(x_0, f^{n(u)}(x_0)\big) \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

so that

$$d\big(f^{n(u)}(u), u\big) \to 0 \quad \text{as } n \to \infty.$$

Hence $f^{n(u)}(u) = u$.

It follows from the remark made earlier that there is a unique $u \in B$ satisfying $f(u) = u$ and $f^n(y_0) \to u$ for each $y_0 \in B$.

The last assertion of the theorem follows as a consequence of the discussion made above.

Acknowledgement. The authors sincerely thank Prof. U. N. Singh, Dean of Faculty of Mathematics, University of Delhi, and Dr. B. D. Sharma, Department of Mathematics, University of Delhi, for their encouragement in the preparation of this paper.

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Reçu par la Rédaction le 15. 4. 1974