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ON HOMOLOGICAL DIMENSIONS OF COCOMMUTATIVE HOPF ALGEBRAS

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Let k be a commutative ring with the unit and let A be an augmented k-algebra with an augmentation $\varepsilon: A \to k$. We denote by $\text{r.gl.dim}A$ (respectively, $\text{l.gl.dim}A$) the right (respectively, left) global dimension of A. The projective dimension of a right (respectively, left) A-module M will be denoted by $\text{r.pd}_A M$ (respectively, $\text{l.pd}_A M$). Finally, the *cohomological dimension* of an algebra A is defined by

$$\dim A = \text{r.pd}_{A^e} A = \text{l.pd}_{A^e} A,$$

where A^e is the enveloping algebra of A (see [3], Chapter IX,3).

Recall that Theorem 6.2 in Chapter X of [3] gives some conditions for

$$\dim A = \text{l.pd}_A k = \text{r.pd}_A k.$$

In particular, these conditions are satisfied for any group algebra (see [3], Chapter X,6) and for the universal enveloping algebra of a Lie algebra (see [3], Chapter XIII, Theorem 5.2).

In this note we show that these conditions are satisfied for any cocommutative Hopf algebra which is projective as a k-module.

Since a cocommutative Hopf algebra may be considered as a group object in the category of cocommutative coalgebras over k, we start (Section 1) with some general facts on group objects in an arbitrary category and then (Section 2) we apply these facts to Hopf algebras.

1. Let C be a category with finite products and with a terminal object 1. A group $G = (G, m, u, s)$ in C is an object G of C together with structural morphisms $m: G\times G \to G$, $u: 1 \to G$, and $s: G \to G$ satisfying the usual conditions (see, e.g., [2], Chapter 4). The morphisms m, u, and s are called the multiplication, unit, and inversion of G, respectively.

Let G be a group in C and let X be a right G-object with the operation $n: X\times G \to X$. If there exists a difference cokernel of n and of the projection $\text{pr}_X: X\times G \to X$, then we denote it by $\pi_X: X \to X/G$. The object X/G is called the *orbit space* of X.

For any object X of C we have a trivial G-object $X\times G$ with the right operation induced by the multiplication of G. It is easy to prove that the orbit space of the trivial G-object $X\times G$ is equal to X and $\pi_{X\times G} = \mathrm{pr}_X$.

If (G, m, u, s) is a group in C, then we define a new group

$$G^{\circ} = (G^{\circ} = G, mt, u, s),$$

where $t\colon G\times G\to G\times G$ is the transposition morphism. The group G° is called the *opposite group* to G. Observe that the inversion s is a group isomorphism of G and G°. Hence we have defined a group morphism

$$E = (\mathrm{id}_G\times s)d\colon G\to G\times G^{\circ},$$

where $d\colon G\to G\times G$ is the diagonal morphism. The group morphism E induces in a natural way the right G-object structure on $G\times G^{\circ}$. The corresponding operation is denoted by N.

LEMMA 1. *The G-object $G\times G^{\circ}$ is isomorphic to the trivial right G-object $G\times G$, the orbit space of $G\times G^{\circ}$ is equal to G, and $\pi_{G\times G^{\circ}} = m$.*

Proof. It is easy to see that the morphism

$$f = \langle m, s\cdot \mathrm{pr}_{G^{\circ}}\rangle\colon G\times G^{\circ}\to G\times G$$

is an isomorphism in C and that the following diagram is commutative:

$$\begin{tikzcd}
G\times G^{\circ}\times G \arrow[r, shift left, "N"] \arrow[r, shift right, "\mathrm{pr}_{G\times G^{\circ}}"'] \arrow[d, "f\times \mathrm{id}_G"] & G\times G^{\circ} \arrow[r, "m"] \arrow[d, "f"] & G \arrow[d, "\mathrm{id}_G"] \\
G\times G\times G \arrow[r, shift left, "\mathrm{id}_G\times m"] \arrow[r, shift right, "\mathrm{pr}_{1,2}"'] & G\times G \arrow[r, "\mathrm{pr}_1"] & G
\end{tikzcd}$$

Since the bottom row is right exact, then so is the top row. This proves the assertions of the lemma.

2. Now let C denote the category of cocommutative coalgebras over a ring k (see [1]). If X and Y are coalgebras, then the k-module $X\otimes_k Y$ has the natural structure of a coalgebra and it is a product of X and Y in C (see [1], Proposition 5.1). A terminal object of C is k. Hence C is a category with finite products. It is obvious that the notions of a group object in C and of a cocommutative Hopf k-algebra coincide.

Let G be a cocommutative Hopf k-algebra. If X is a right G-object with the operation $n\colon X\otimes_k G\to X$, then, in particular, X is a right G-module with n as the structure morphism.

Let I denote the kernel of the counit $e\colon G\to k$.

LEMMA 2. *Let X be a G-object. Then* $\ker\pi_X = X\cdot I$, *and there is a k-module isomorphism* $X/G = X\otimes_G k$.

Proof. It is easy to see that the category C is cocomplete and the forgetful functor from C to the category of k-modules preserves colimits. Hence we have the equality

$$\ker \pi_X = \operatorname{im}(n - \mathrm{pr}_X).$$

Moreover,

$$\mathrm{pr}_X(x \otimes g) = x \cdot e(g) \quad \text{for } x \otimes g \in X \otimes_k G.$$

The assertions of the lemma follow immediately.

As a consequence of Lemmas 1 and 2 we obtain

COROLLARY. (i) $\ker m = G^e \cdot I$, *where* $G^e = G \otimes_k G^o$.

(ii) *If G is projective as a k-module, then G^e is projective as a right G-module.*

The following result is a consequence of Theorem 6.2 in [3], Chapter X, and of the previous results.

THEOREM. *Let G be a cocommutative Hopf k-algebra which is projective as a k-module. Then*

$$\dim G = \mathrm{l.pd}_G k = \mathrm{r.pd}_G k.$$

Moreover, if k is semi-simple, then

$$\dim G = \mathrm{l.gl.dim} G = \mathrm{r.gl.dim} G.$$

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