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SPECTRAL RADIUS AND SEMINORMS IN FINITE-DIMENSIONAL ALGEBRAS

BY

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It is well known that if $\mathscr{A}$ is a Banach algebra with identity, with associated norm $\|\cdot\|$, then

$$\lim_n \|A^n\|^{1/n} = r(A),$$

the spectral radius of A, for all $A \in \mathscr{A}$ (see [2], p. 179). Wimmer proved [4] that, for the algebra $\mathscr{M}_k$ of $k \times k$ complex matrices, and for the trace linear functional Tr on $\mathscr{M}_k$,

$$\limsup_n |\mathrm{Tr}(A^n)|^{1/n} = r(A) \quad \text{for all } A \in \mathscr{M}_k. \tag{1}$$

Wimmer has also raised the problem of characterizing those seminorms ϱ on $\mathscr{M}_k$ which satisfy

$$\lim_n \varrho(A^n)^{1/n} = r(A) \quad \text{for all } A \in \mathscr{M}_k. \tag{2}$$

This problem will not be solved in this note. Here we solve an easier problem in a more general setting, a natural generalization of Wimmer's result. Specifically, we give a necessary and sufficient condition for a seminorm ϱ on an algebra $\mathscr{A}$, finite dimensional over the complex numbers $\boldsymbol{C}$, and with identity, to satisfy

$$\limsup_n \varrho(A^n)^{1/n} = r(A) \quad \text{for all } A \in \mathscr{A}. \tag{3}$$

The characterization is very satisfying by its simplicity; its usefulness will be given a little promotion here by an application to $\mathscr{M}_k$.

Before the result is stated, some facts, basic to its proof, about finite-dimensional algebras with identity, and seminorms on finite-dimensional vector spaces, will be summarized.

If $\mathscr{A}$ is an algebra with identity, finite dimensional over $\boldsymbol{C}$, and $A \in \mathscr{A}$, then, because $\mathscr{A}$ is finite dimensional, there is a unique, monic polynomial $m_A(x) \in \boldsymbol{C}[x]$ such that $m_A(A) = 0$, and $p(A) = 0$, $p(x) \in \boldsymbol{C}[x]$ imply

$m_A(x) \mid p(x)$. Observe that, clearly, A is invertible if and only if $m_A(0) \neq 0$. It follows easily that the elements of the spectrum of A are precisely the roots of $m_A(x)$. Consequently, the spectral radius and, indeed, the spectrum of $A \in \mathscr{A}$ are the same as the spectral radius and the spectrum of A considered as an element of any subalgebra of $\mathscr{A}$, with identity, that happens to contain A.

Suppose that E is a vector space over C, and ϱ and τ are two seminorms on E; $\varrho \prec \tau$ means that there is a number $K > 0$ such that $\varrho(x) \leqslant K\tau(x)$ for all $x \in E$. If $\varrho \prec \tau$ and $\tau \prec \varrho$, then ϱ and τ are said to be *equivalent*. Set

$$N(\varrho) = \{x \in E;\ \varrho(x) = 0\}.$$

$N(\varrho)$ is a subspace of E. Clearly, $\varrho \prec \tau$ implies $N(\tau) \subseteq N(\varrho)$. If E is finite dimensional, then the converse is true: $N(\tau) \subseteq N(\varrho)$ implies $\varrho \prec \tau$.

For the proof observe that if $N(\tau) \subseteq N(\varrho)$, then τ induces a norm, and ϱ a seminorm, on $E/N(\tau)$. It is well known that any seminorm on a finite-dimensional normed space is continuous. Indeed, this fact follows easily from the well-known equivalence of norms on finite-dimensional spaces, a proof of which may be found in [3], p. 95. Returning to E from $E/N(\tau)$, we conclude that $\varrho \prec \tau$ follows when E is finite dimensional or, more generally, when $N(\tau)$ is finite codimensional in E.

Now suppose that $E = \mathscr{A}$, a finite-dimensional algebra with identity. If $A \in \mathscr{A}$ and

$$\limsup_n \varrho(A^n)^{1/n} = r(A),$$

and ϱ and τ are equivalent, then the same equation holds with ϱ replaced by τ. Thus, whether or not ϱ satisfies (3) is determined by the nature of $N(\varrho)$.

Finally, for $A \in \mathscr{A}$, to show $\limsup_n \varrho(A^n)^{1/n} = r(A)$, it suffices to prove that

$$\limsup_n \varrho(A^n)^{1/n} \geqslant r(A).$$

The reason is that $\mathscr{A}$ may be represented as a subalgebra of $\mathscr{M}_k$ for some positive integer k (see [2], p. 180). There is, therefore, a Banach algebra norm $\|\cdot\|$ on $\mathscr{A}$; $\{0\} = N(\|\cdot\|) \subseteq N(\varrho)$, so $\varrho \prec \|\cdot\|$, whence

$$\limsup_n \varrho(A^n)^{1/n} \leqslant \lim_n \|A^n\|^{1/n} = r(A).$$

THEOREM 1. *Suppose that $\mathscr{A}$ is an algebra with identity, finite dimensional over C, and ϱ is a seminorm on $\mathscr{A}$. Then ϱ satisfies* (3) *if and only if $N(\varrho)$ contains no non-zero idempotent.*

Proof. If $A \in N(\varrho)$, $A \neq 0$, and $A^2 = A$, then $A^n = A \in N(\varrho)$ for all positive integers n. Thus

$$\limsup_n \varrho(A^n)^{1/n} = 0 \neq 1 = r(A).$$

Now suppose that $N(\varrho)$ contains no non-zero idempotent, and let $A \in \mathscr{A}$. We want to show that $\limsup_n \varrho(A^n)^{1/n} = r(A)$. It suffices to prove that

$$\limsup_n \varrho(A^n)^{1/n} \geqslant r(A),$$

and that this inequality holds with ϱ replaced by the restriction of ϱ to the subalgebra of $\mathscr{A}$ generated by A and the identity. For simplicity, we will henceforward assume that $\mathscr{A}$ is generated by A and the identity; that is, every element of $\mathscr{A}$ is a polynomial in A.

Then the map $P(x) \to P(A)$ is an algebra homomorphism of $C[x]$ onto $\mathscr{A}$. Consequently, $\mathscr{A}$ is isomorphic to $C[x]/(m(x))$, with $m(x) = m_A(x)$ being the minimal polynomial of A, and $(m(x))$ the ideal generated by it.

Let $a_1, \ldots, a_t$ be the distinct roots of $m(x)$,

$$m(x) = \prod_{j=1}^{t} (x - a_j)^{s_j}.$$

Suppose that $|a_1| \leqslant \ldots \leqslant |a_t| = r(A)$. Then $C[x]/(m(x))$ is an algebra isomorphic to the direct sum

$$\mathop{\mathsf{X}}_{j=1}^{t} C[x]/((x-a_j)^{s_j})$$

(using the Chinese Remainder Theorem applied to $C[x]$) by the map which sends an element $P(x) + (m(x))$ of $C[x]/(m(x))$ onto

$$\big(P(x) + ((x-a_1)^{s_1}), \ldots, P(x) + ((x-a_t)^{s_t})\big).$$

Clearly, the cosets represented by $1, x - a_j, \ldots, (x-a_j)^{s_j - 1}$ form a basis of $C[x]/((x-a_j)^{s_j})$, considered as a vector space over C. Let U_{ij} denote the coset of $(x-a_j)^{i-1}$ in $C[x]/((x-a_j)^{s_j})$, and let B_{ij} denote the element of $\mathscr{A}$ corresponding to $(0, \ldots, U_{ij}, 0, \ldots)$ in the correspondence of $\mathscr{A}$ to

$$\mathop{\mathsf{X}}_{j=1}^{t} C[x]/((x-a_j)^{s_j}).$$

(By $(0, \ldots, U_{ij}, 0, \ldots)$ is meant the element of the product with U_{ij} in the j-th position, and zeroes elsewhere.) The B_{ij} $(i = 1, \ldots, s_j;\ j = 1, \ldots, t)$ constitute a basis of $\mathscr{A}$. Note that

$$A = \sum_{j=1}^{t} (a_j B_{1j} + B_{2j}),$$

since $x = a_j \cdot 1 + (x - a_j)$ for each j. Note further that B_{1j} $(j = 1, \ldots, t)$ is a non-zero idempotent.

If z and a are complex numbers such that $z \neq a$, then

$$(z-x)\sum_{i=1}^{s}(z-a)^{-i}(x-a)^{i-1} = \big((z-a)-(x-a)\big)\sum_{i=1}^{s}(z-a)^{-i}(x-a)^{i-1}$$
$$= 1-(z-a)^{-s}(x-a)^{s} \equiv 1 \pmod{(x-a)^{s}}.$$

That is, the inverse of $z-x$ in $C[x]/\big((x-a)^s\big)$, if $z-x$ has an inverse, is

$$\sum_{i=1}^{s}(z-a)^{-i}(x-a)^{i-1}.$$

It follows (if we let I represent the identity in $\mathscr{A}$) that

$$(zI-A)^{-1} = \sum_{j=1}^{t}\sum_{i=1}^{s_j}(z-a_j)^{-i}B_{ij}$$

for any z such that $z \neq a_j$ $(j = 1, \ldots, t)$. On the other hand, for z sufficiently large,

$$(zI-A)^{-1} = \sum_{n=0}^{\infty} z^{-(n+1)}A^n$$

(see [2], p. 171) with the convergence of the series on the right-hand side taking place in any norm whatever on $\mathscr{A}$.

If $t = 1$ and $a_1 = 0$, then A is nilpotent, and $\varrho(A^n)^{1/n} = 0 = r(A)$ for all n sufficiently large. So suppose that $r(A) = |a_t| > 0$. Since $N(\varrho)$ contains no non-zero idempotents, $B_{1t} \notin N(\varrho)$. Consequently, there is a linear functional $f \in N(\varrho)^0$ ($N(\varrho)^0$ is the annihilator of $N(\varrho)$ in $\mathscr{A}'$, the dual of $\mathscr{A}$) such that $f(B_{1t}) \neq 0$. Then

$$g(z) = f\big((zI-A)^{-1}\big) = \sum_{j=1}^{t}\sum_{i=1}^{s_j}(z-a_j)^{-i}f(B_{ij})$$

has clearly a pole at a_t, since it has a non-zero residue $f(B_{1t})$ there.

Consequently, the radius of convergence of the power series

$$g\left(\frac{1}{w}\right) = \sum_{n=0}^{\infty} w^{n+1}f(A^n)$$

is less than or equal to $|a_t|^{-1} = r(A)_{\mathbf{A}}^{-1}$, which, by Hadamard's convergence formula, implies that

$$\limsup_{n} |f(A^n)|^{1/n} \geqslant r(A).$$

The seminorm τ defined by $\tau(B) = |f(B)|$ for all $B \in \mathscr{A}$ satisfies $N(\varrho) \subseteq N(\tau)$, since $f \in N(\varrho)^0$. Therefore, $\tau \prec \varrho$, so

$$\limsup_n \varrho(A^n)^{1/n} \geqslant \limsup_n \tau(A^n)^{1/n} \geqslant r(A).$$

COROLLARY 1. *Suppose that $\mathscr{A}$ is a subalgebra of $\mathscr{M}_k$, with or without identity, and ϱ is a seminorm on $\mathscr{A}$. Then the following statements are equivalent:*

(a) *ϱ satisfies* (3);

(b) *$N(\varrho)$ contains no non-zero idempotent;*

(c) *ϱ has an extension to all of $\mathscr{M}_k$, satisfying* (3) *with $\mathscr{A}$ replaced by $\mathscr{M}_k$.*

Remark. Since any finite-dimensional algebra, with or without identity, may be considered as a subalgebra of $\mathscr{M}_k$ for some k (in case where the algebra has no identity, adjoin an identity and embed the augmented algebra in $\mathscr{M}_k$ for some k), the equivalence of (a) and (b) extends effectively Theorem 1 to finite-dimensional algebras without identity.

Proof. Clearly, (c) implies (a), and also (a) implies (b) by the first part of the proof of Theorem 1. We show that (b) implies (c). Let $\|\cdot\|$ be any norm on $\mathscr{M}_k$, and let P be any projection of $\mathscr{M}_k$ onto $\mathscr{A}$. Write

$$\bar{\varrho}(A) = \|A - P(A)\| + \varrho\big(P(A)\big) \quad \text{for all } A \in \mathscr{M}_k.$$

Clearly, $\bar{\varrho}$ is a seminorm on $\mathscr{M}_k$, an extension of ϱ, and $N(\varrho) = N(\bar{\varrho})$, so $N(\bar{\varrho})$ contains no non-zero idempotent. Thus statement (c) follows from Theorem 1 applied to $\mathscr{M}_k$ and $\bar{\varrho}$.

The next corollary requires some additional terminology and notation. For $A \in \mathscr{M}_k$, the conjugate transpose or adjoint of A will be denoted by A^*. If $A, B \in \mathscr{M}_k$, the Hilbert-Schmidt inner product of A and B is the scalar

$$[A, B] = \sum_{i=1}^{k} \sum_{j=1}^{k} A_{ij} \bar{B}_{ij}.$$

Clearly, $[\cdot, \cdot]$ is a complex inner product on $\mathscr{M}_k$; indeed, it is just the usual inner product on $\mathscr{M}_k$ considered in an obvious way as C^{k^2}.

The positive integer k will be fixed. For $S \subseteq \{1, \ldots, k\} \times \{1, \ldots, k\}$, $\chi_S \in \mathscr{M}_k$ will be the matrix with 1 at every entry indexed by an ordered pair in S, and zeroes elsewhere. Note that, for any $B \in \mathscr{M}_k$,

$$[\chi_S, B] = \sum_{(i,j)\in S} \bar{B}_{ij}.$$

An upper diagonal will be a set of ordered pairs of the form

$$\{(i, p+i-1);\ i = 1, \ldots, k-p+1,\ p = 1, \ldots, k\}.$$

A lower diagonal will be a set of ordered pairs of the form

$$\{(p+i-1, i);\ i = 1, \ldots, k-p+1,\ p = 1, \ldots, k\}.$$

The main diagonal, $\{(1,1), \ldots, (k,k)\}$, is thus both upper and lower. A diagonal block is the Cartesian product of any non-empty block of consecutive integers, from $\{1, \ldots, k\}$, with itself.

It will be useful to note that

$$[A, B] = [B^*, A^*] \quad \text{and} \quad [AB, C] = [C^*A, B^*] \quad \text{for any } A, B, C \in \mathscr{M}_k.$$

It follows that if $A, B, P \in \mathscr{M}_k$, and P is invertible, then

$$[PAP^{-1}, B] = [A, P^*B(P^*)^{-1}].$$

COROLLARY 2. *Suppose that $B_1, \ldots, B_t \in \mathscr{M}_k$. The following statements are equivalent:*

(a) *for each $A \in \mathscr{M}_k$,*

$$\limsup_n \max_{m=1,\ldots,t} \|[A^n, B_m]\|^{1/n} = r(A);$$

(b) *there is no non-zero idempotent $A \in \mathscr{M}_k$ such that $[A, B_m] = 0$ for each $m = 1, \ldots, t$;*

(c) *for each invertible $P \in \mathscr{M}_k$ and each non-empty subset S of the main diagonal, there exists $m \in \{1, \ldots, t\}$ such that*

$$[\chi_S, PB_mP^{-1}] \neq 0;$$

(d) ((d′), (d″)) (1) *for each invertible $P \in \mathscr{M}_k$, and each non-empty list $J_1, \ldots, J_q$ of mutually disjoint diagonal blocks, there exist $m \in \{1, \ldots, t\}$ and an (upper, lower) diagonal D such that, letting*

$$S = D \cap \Big(\bigcup_{i=1}^{q} J_i\Big),$$

we have $[\chi_S, PB_mP^{-1}] \neq 0$.

Proof. Define ϱ on $\mathscr{M}_k$ by

$$\varrho(A) = \max_{m=1,\ldots,t} \|[A, B_m]\|.$$

Clearly, ϱ is a seminorm on $\mathscr{M}_k$, and

$$N(\varrho) = \{B_1, \ldots, B_t\}^{\perp} = \{A;\ [A, B_m] = 0,\ m = 1, \ldots, t\}.$$

Thus (a) and (b) are equivalent by Theorem 1.

(b) implies (c). If p is invertible, and S is a non-empty subset of the main diagonal, then $P^*\chi_S(P^*)^{-1}$ is a non-zero idempotent. Thus, by (b), there exists $m \in \{1, \ldots, t\}$ such that

$$[P^*\chi_S(P^*)^{-1}, B_m] = [\chi_S, PB_mP^{-1}] \neq 0.$$

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(1) The different statements (d) are obtained by writing "upper" or "lower" or nothing in front of the second occurrence of the word "diagonal".

(c) implies any of statements (d). Given P and $J_1, \ldots, J_q$ as in (d), let D be the main diagonal. Then

$$D \cap \left(\bigcup_{i=1}^{q} J_i\right) = S$$

is a non-empty subset of the main diagonal, whence the desired conclusion follows from (c).

Among statements (d), each of the first two implies the third, the one in which the type of diagonal whose existence is asserted is unqualified by "upper" or "lower". We show that this weakest statement of the three implies (c). Given

$$S = \{(s_1, s_1), \ldots, (s_q, s_q)\}, \quad 1 \leqslant s_1 < \ldots < s_q \leqslant k,$$

a non-empty subset of the main diagonal, and an invertible $P \in \mathcal{M}_k$, set $J_i = \{(s_i, s_i)\}$, $i = 1, \ldots, q$. Then $J_1, \ldots, J_q$ are mutually disjoint, non-empty diagonal blocks. The diagonal D whose existence is asserted in (d″) must be the main diagonal, in view of the full conclusion of (d″), since no other diagonal intersects $\bigcup_{i=1}^{q} J_i$. Then $S = D \cap \left(\bigcup_{i=1}^{q} J_i\right)$, and the conclusion of (c) follows.

It may be of interest here to note that (d) and (d′) are easily seen to be equivalent by the fact that each $A \in \mathcal{M}_k$ is similar to its transpose (see [1], p. 115, Problem 8), but that the equivalence of all three statements (d) is not trivially obvious.

The proof will be completed if we show that (c) implies (b). If A is non-zero idempotent, then, for some invertible P and some non-empty subset S of the main diagonal, $A = P\chi_S P^{-1}$. Then by (c) there exists $m \in \{1, \ldots, t\}$ such that

$$0 \neq [\chi_S, P^* B_m (P^*)^{-1}] = [A, B_m].$$

COROLLARY 3. *If* $B \in \mathcal{M}_k$ *and*

$$\limsup_n |[A^n, B]|^{1/n} = r(A) \quad \textit{for each } A \in \mathcal{M}_k,$$

then B *is a non-zero scalar multiple of the identity matrix.*

Proof. If some entry of B off the main diagonal is non-zero, say $B_{ij} \neq 0$ with $i \neq j$, set

$$A = \chi_{\{(i,i)\}} - \frac{\bar{B}_{ii}\cdot}{B_{ij}} \chi_{\{(i,j)\}}.$$

Then A is a non-zero idempotent, and $[A, B] = 0$. Thus B is a diagonal matrix. If any two main diagonal entries of B are unequal, then B is similar to a non-diagonal matrix. Hence the construction above, together with the properties of $[\cdot, \cdot]$, again yields a non-zero idempotent A such that $[A, B] = 0$. Thus B is a scalar multiple of I, and the scalar is surely not zero.

Corollary 4 asserts that, up to equivalence, the absolute value of the trace is the only possible seminorm on $\mathcal{M}_k$ with null space of codimension one that could satisfy (3). That $|\mathrm{Tr}(\cdot)|$ does indeed satisfy (3) was Wimmer's result, and it follows also from Corollary 2. It is reasonable to ask if $|\mathrm{Tr}(\cdot)|$ is the weakest seminorm on $\mathcal{M}_k$, with respect to the ordering $\prec$, modulo equivalence, satisfying (3). Equivalently, if $B_1, \ldots, B_t \in \mathcal{M}_k$ satisfy (a) of Corollary 2, is it necessarily the case where I lies in the linear span of $B_1, \ldots, B_t$? The answer is no.

Example 1. Set

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Clearly, $I \notin \mathrm{sp}(B_1, B_2)$, and it is claimed that B_1 and B_2 satisfy (a) of Corollary 2. We will check the claim directly with the use of (c) of Corollary 2. Suppose that

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible; we may also assume that $\det P = ad - bc = 1$. Then

$$P^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad PB_1P^{-1} = \begin{bmatrix} ad-2bc & ab \\ -cd & -bc+2ad \end{bmatrix},$$

$$PB_2P^{-1} = \begin{bmatrix} bd & -b^2 \\ d^2 & -bd \end{bmatrix}.$$

The sum of the main diagonal entries of PB_1P^{-1}, i.e. the trace of PB_1P^{-1}, is, of course, 3, so to show that B_1 and B_2 satisfy (c) of Corollary 2 it suffices to prove that it is impossible for the (1, 1)-entries of PB_1P^{-1} and PB_2P^{-1} to be simultaneously zero, and the same holds for (2, 2).

If $bd = 0$, then either $b = 0$ or $d = 0$. If $b = 0$, then $ad - bc \neq 0$ implies $ad = ad - 2bc \neq 0$. If $d = 0$, then $ad - bc \neq 0$ implies $-2bc = ad - -2bc \neq 0$. The argument about the (2, 2)-entries is similar.

COROLLARY 4. *If $B_1, \ldots, B_t \in \mathcal{M}_k$ and the span of all the columns (or rows) of all the $B_1, \ldots, B_t$ is a proper subspace of C^k, then $B_1, \ldots, B_t$ do not satisfy* (a) *of Corollary* 2.

Proof. Let V be a proper subspace of C^k to which the columns of $B_1, \ldots, B_t$ are confined, and let Q be the matrix representing (with respect to the usual basis of C^k) the orthogonal projection of C^k (with respect to the usual inner product on C^k) onto the orthogonal complement of V. Then Q is a non-zero idempotent. The columns of Q are in $V^\perp$, so, clearly, $[Q, B_m] = 0$ for each $m = 1, \ldots, t$.

Before leaving Theorem 1 and its consequences, it may be instructive to observe in what way Wimmer's original result is a special case of Theorem 1 with $\mathscr{A} = \mathscr{M}_k$. Wimmer's result may be deduced from Theorem 1 by using the fact that an idempotent matrix with trace zero must be the zero matrix. Alternatively, this little fact, an easy exercise in elementary linear algebra, may grandly be deduced from Wimmer's result and Theorem 1.

Infinite-dimensional analogues. For a vector space E, and a subspace F of E, let Q_F^E denote the canonical map of E onto E/F. If E is equipped with norm $\|\cdot\|$, and F is closed, $\|Q_F^E(\cdot)\|$ will denote the quotient norm on E/F.

It might be expected that the simplest extension of the inquiry with which this paper started would be effected by replacing finite-dimensional algebras by arbitrary Banach algebras, and by requiring the seminorms under investigation to be continuous. There is, however, a simpler question to be asked: for which closed subspaces N of a complex Banach algebra $\mathscr{A}$ with identity, is it the case that

(4) $$r(A) = \lim_n \|Q_N^{\mathscr{A}}(A^n)\|^{1/n} \quad \text{for all } A \in \mathscr{A}$$

or

(5) $$r(A) = \limsup_n \|Q_N^{\mathscr{A}}(A^n)\|^{1/n} \quad \text{for all } A \in \mathscr{A}?$$

If ϱ is a continuous seminorm on $\mathscr{A}$, then $N = N(\varrho)$ is a closed subspace of $\mathscr{A}$, and $\varrho \prec \|Q_N^{\mathscr{A}}(\cdot)\|$. It follows that if ϱ satisfies (2) or (3), then N satisfies (4) or (5), respectively. The converse implications, however, do not work.

Example 2. Let $\mathscr{A} = \mathscr{L}_1$ be the space of absolutely summable sequences of complex numbers with the usual norm and with the convolution multiplication

$$a \cdot b = \Big(\sum_{k=0}^{n} a_k b_{n-k}\Big)_{n=0}.$$

Define ϱ on $\mathscr{L}_1$ by

$$\varrho(a) = \sum_{k=0}^{\infty} 2^{-k}|a_k|.$$

Then ϱ is surely a continuous seminorm on $\mathscr{A}$, and $N(\varrho) = \{0\}$, so (4) holds with $N = N(\varrho)$. Set $e_1 = (0, 1, 0, 0, \ldots)$. Then $\varrho(e_1^n)^{1/n} = \frac{1}{2}$ for all n, whereas $r(e_1) = 1$, so (3) and, consequently, (2) fail to hold.

In modest contribution to the unravelling of the problems indicated by (4), (5), and, for finite-dimensional algebras, (2), one proposition and some corollaries will be presented here. The proposition is well known, although the author would not know to whom to attribute it.

The Jacobson radical J of a Banach algebra $\mathscr{A}$ with identity I is the set of elements A such that $I+BA$ has a two-sided inverse for all $B \in \mathscr{A}$. It is also the intersection of all maximal left ideals of $\mathscr{A}$, and of all maximal right ideals, and is, therefore, a two-sided ideal (see [2], p. 161 and 162). It is closed, since any maximal left ideal in a Banach algebra with identity is closed. Thus $\mathscr{A}/J$ is itself a Banach algebra with the norm $\|Q_J^{\mathscr{A}}(\cdot)\|$.

PROPOSITION 1. *Suppose that $\mathscr{A}$ and J are as in the lines above. For all $A \in \mathscr{A}$, the spectrum of A and that of $Q_J^{\mathscr{A}}(A)$ are the same.*

Proof. Let $\sigma(A)$ denote the spectrum of A, and set $Q = Q_J^{\mathscr{A}}$. If $\lambda I - A$ is invertible (in $\mathscr{A}$), then so is $\lambda Q(I) - Q(A)$, whence $\sigma(Q(A)) \subseteq \sigma(A)$. On the other hand, if $\lambda Q(I) - Q(A)$ is invertible in $\mathscr{A}/J$, then there exists $B \in \mathscr{A}$ such that

$$B(\lambda I - A) - I = C_1 \in J \quad \text{and} \quad (\lambda I - A)B - I = C_2 \in J.$$

Now, $I + C_1$ has a two-sided inverse in $\mathscr{A}$ and, in particular, a left inverse, so $\lambda I - A$ has a left inverse. Similarly, $\lambda I - A$ has a right inverse, which completes the proof.

COROLLARY 5. *If N is a subspace of J (not necessarily closed), then*

$$r(A) = \lim_n \|Q_N^{\mathscr{A}}(A^n)\|^{1/n} \quad \textit{for all } A \in \mathscr{A}.$$

The result follows from Proposition 1 and the inequalities

$$\|Q_J^{\mathscr{A}}(A)\| \leqslant \|Q_N^{\mathscr{A}}(A)\| \leqslant \|A\| \quad \text{for all } A \in \mathscr{A}.$$

COROLLARY 6. *If $\mathscr{A}$ is a finite-dimensional algebra with identity, ϱ a seminorm on $\mathscr{A}$, and $N(\varrho)$ a subspace of a nilpotent left (or right) ideal in $\mathscr{A}$ (or, which is the same thing in the finite-dimensional case, a left or right ideal all of which elements are nilpotent), then*

$$\lim_n \varrho(A^n)^{1/n} = r(A) \quad \textit{for all } A \in \mathscr{A}.$$

Proof. Suppose that $N(\varrho) = N \subseteq \bar{N}$ is a nilpotent left ideal. If $B \in \bar{N}$ and $C \in \mathscr{A}$, then $CB \in \bar{N}$, so CB is nilpotent. Therefore, $I + CB$ is invertible. Since C is arbitrary, $B \in J$, the Jacobson radical of $\mathscr{A}$. Now the result follows from Corollary 5 and from the fact that ϱ and $\|Q_N^{\mathscr{A}}(\cdot)\|$ are equivalent for any Banach algebra norm $\|\cdot\|$ on $\mathscr{A}$.

Proposition 1 has the following alternative proof in the finite-dimensional case. In that case, J is a two-sided nilpotent ideal. For $A \in \mathscr{A}$, let $m(x)$ be the minimal polynomial of A in $\mathscr{A}$, and $\tilde{m}(x)$ the minimal polynomial of $Q(A)$ in $\mathscr{A}/J$. Then, surely, $\tilde{m}(x)$ divides $m(x)$. On the other hand, since $\tilde{m}(A)$ is nilpotent, $m(x)$ divides some integral power of $\tilde{m}(x)$. Thus the distinct linear factors of the two polynomials are the same.

It should be noted that, in Proposition 1 and its corollaries, $\mathscr{A}$ is assumed to be a complex algebra.

Some problems. (a) Does Theorem 1 remain true if $\mathscr{A}$ is taken to be a real algebra, say a subalgebra with identity of the $k \times k$ real matrices, with $r(A)$ still denoting the spectral radius of A as a complex matrix? (**P 1039**)

(b) Characterize those subalgebras $\mathscr{A}$ of $\mathscr{M}_k$, with identity, for which

$$B \in \mathscr{A} \quad \text{and} \quad \limsup_n |[A^n, B]|^{1/n} = r(A) \quad \text{for all } A \in \mathscr{A}$$

imply that B is a non-zero scalar times the identity. (Corollary 3 asserts that $\mathscr{M}_k$ itself has this property. Surely, the one-dimensional algebra consisting of scalar multiples of the identity does as well.) (**P 1040**)

(c) Characterize those subalgebras $\mathscr{A}$ of $\mathscr{M}_k$ for which, if ϱ is a seminorm on $\mathscr{A}$ and

$$\limsup_n \varrho(A^n)^{1/n} = r(A) \quad \text{for all } A \in \mathscr{A},$$

then it must be the case that $|\mathrm{Tr}(\cdot)| \prec \varrho$ on $\mathscr{A}$. (Example 5 shows that $\mathscr{M}_k$ itself does not have this property. The algebra generated by the identity alone does trivially.) (**P 1041**)

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