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A THEOREM ON THE ABSOLUTE RIESZ SUMMABILITY FACTORS OF A FOURIER SERIES

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1.1. Definition. Let $\lambda = \lambda(\omega)$ be a differentiable and monotonic increasing function of ω in (K, ∞), K being a positive constant, and let $\lambda(\omega)$ tend to infinity with ω. Let $\sum a_n$ be a given infinite series, and let

$$C_r(\omega) = \sum_{n\leqslant\omega}\{\lambda(\omega)-\lambda(n)\}^r a_n, \qquad r>0.$$

The series $\sum a_n$ is said to be *summable* $|R, \lambda, r|$, $r>0$, if the integral

$$\int_A^\infty \left| d\frac{C_r(\omega)}{\{\lambda(\omega)\}^r}\right|,$$

where A is a finite positive number, is convergent (cf. [5] and [6]). Now, for $r>0$ and $m<\omega<m+1$,

$$\frac{d}{d\omega}\left[\frac{C_r(\omega)}{\{\lambda(\omega)\}^r}\right] = \frac{r\lambda'(\omega)}{\{\lambda(\omega)\}^{r+1}}\sum_{n\leqslant\omega}\{\lambda(\omega)-\lambda(n)\}^{r-1}\lambda(n)a_n.$$

Hence $\sum a_n$ is summable $|R, \lambda, r|$, $r>0$, if

$$\int_A^\infty \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^{r+1}}\left|\sum_{n\leqslant\omega}\{\lambda(\omega)-\lambda(n)\}^{r-1}\lambda(n)a_n\right| d\omega < \infty.$$

Summability $|R, \lambda, 0|$, by definition, is the same as absolute convergence.

1.2. Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without any loss of generality, the constant term in the Fourier series of $f(t)$ can be taken to be zero, so that

$$f(t) \sim \sum_1^\infty (a_n\cos nt + b_n\sin nt) = \sum_1^\infty A_n(t) \tag{1.2.1}$$

and

(1.2.2)
$$\int_{-\pi}^{\pi} f(t)\,dt = 0.$$

Here and elsewhere $\log_2 n$ means $\log\log n$; and k is a suitable constant as required by the hypotheses of the theorem. Throughout the present paper we use the following notation:

$$\varphi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)\};$$

$$\theta(\omega, t) = \sum_{3\leqslant n\leqslant\omega} \exp[(\log n)^{\Delta}](\log_2 n)^{\varepsilon-1}\cos nt;$$

$$\eta(\omega, t) = \sum_{3\leqslant n\leqslant\omega} \exp[(\log n)^{\Delta}](\log_2 n)^{\varepsilon-1}n^{-1}\sin nt;$$

$$g(\omega, t) = \int_0^t \big(\log_2(k/u)\big)^{-\varepsilon}\theta(\omega, u)\,du;$$

$$h(\omega, t) = \int_t^{\pi} \big(\log_2(k/u)\big)^{-\varepsilon}\theta(\omega, u)\,du;$$

$$\Phi_a(t) = \frac{1}{\Gamma(a)}\int_0^t (t-u)^{a-1}\varphi(u)\,du \qquad (a>0);$$

$$\Phi_0(t) = \varphi(t);$$

$$\varPhi_a(t) = \Gamma(a+1)t^{-a}\Phi_a(t) \qquad (a\geqslant 0).$$

Concerning the absolute Riesz summability factors of a Fourier series, Mohanty and Misra [4] proved:

If $\varPhi_a(t)$ is of bounded variation in $(0, \pi)$, then the series $\sum_{2}^{\infty} An/\log n$ is summable $|R, \exp[(\log w)^{\Delta}], 1|$, where $\Delta = 1+1/a$ and $0 < a < 1$.

The author [1] studied the summability of the series $\sum_{2}^{\infty} An/(\log n)^{\zeta}$, $\zeta > 0$, whenever $\varPhi_1(t)$ is of bounded variation in $(0, \pi)$.

1.3. The object of the present paper is to prove the following

THEOREM. *If* $\varphi(t)\big(\log_2(k/t)\big)^{\varepsilon}$, $\varepsilon \geqslant 1$, *is of bounded variation in* $(0, \pi)$, *then the series*

$$\sum_{3}^{\infty} (\log_2 n)^{\varepsilon-1} A_n(x)$$

is summable

$$|R, \exp[(\log\omega)^{\Delta}], 1|$$

for any positive Δ.

1.4. For the proof of our theorem, we require the following inequalities (cf. [1]-[3]):

(1.4.1) $$\theta(\omega, t) = O\left\{\frac{\omega \exp[(\log\omega)^{\Delta}](\log_2\omega)^{\varepsilon-1}}{(\log\omega)^{\Delta-1}}\right\};$$

(1.4.2) $$\sum_{3\leqslant n\leqslant\omega} n^{-1}\exp[(\log n)^{\Delta}](\log_2 n)^{\varepsilon-1} = O\left\{\frac{\exp[(\log\omega)^{\Delta}](\log_2\omega)^{\varepsilon-1}}{(\log\omega)^{\Delta-1}}\right\};$$

(1.4.3) $$\eta(\omega, t) = O\{\exp[(\log\omega)^{\Delta}](\log_2\omega)^{\varepsilon-1}\omega^{-1}t^{-1}\};$$

(1.4.4) $$g(\omega, t) = O\left\{\frac{\omega t \exp[(\log\omega)^{\Delta}](\log_2\omega)^{\varepsilon-1}\big(\log_2(k/t)\big)^{-\varepsilon}}{(\log\omega)^{\Delta-1}}\right\};$$

(1.4.5) $$g(\omega, t) = O\left\{\frac{\exp[(\log\omega)^{\Delta}](\log_2\omega)^{\varepsilon-1}\big(\log_2(k/t)\big)^{-\varepsilon}}{(\log\omega)^{\Delta-1}}\right\};$$

(1.4.6) $$h(\omega, t) = O\{\exp[(\log\omega)^{\Delta}](\log_2\omega)^{\varepsilon-1}\omega^{-1}t^{-1}\}.$$

Proof of (1.4.1). Let $m-1\leqslant\omega<m$. Then

$$\theta(\omega, t) = \sum_{n=3}^{m}\exp[(\log n)^{\Delta}](\log_2 n)^{\varepsilon-1}\cos nt.$$

Hence

$$|\theta(\omega, t)| \leqslant \sum_{3}^{m}\exp[(\log n)^{\Delta}](\log_2 n)^{\varepsilon-1}.$$

Now, we have

$$\sum_{3}^{m}\exp[(\log n)^{\Delta}](\log_2 n)^{\varepsilon-1} < \int_{3}^{\omega}\exp[(\log x)^{\Delta}](\log_2 x)^{\varepsilon-1}dx + $$
$$+\exp[(\log m)^{\Delta}](\log_2 m)^{\varepsilon-1}$$
$$= O\left\{\frac{\omega(\log_2\omega)^{\varepsilon-1}}{(\log\omega)^{\Delta-1}}\int_{3}^{\omega}\frac{\exp[(\log x)^{\Delta}](\log x)^{\Delta-1}}{x}dx\right\}+$$
$$+O\{\exp[(\log\omega)^{\Delta}](\log_2\omega^{\varepsilon-1})\}$$
$$= O\left\{\frac{\omega(\log_2\omega)^{\varepsilon-1}\exp[(\log\omega)^{\Delta}]}{(\log\omega)^{\Delta-1}}\right\}.$$

Hence

$$\theta(\omega, t) = O\left\{\frac{\omega(\log_2\omega)^{\varepsilon-1}\exp[(\log\omega)^{\Delta}]}{(\log\omega)^{\Delta-1}}\right\}.$$

Proof of (1.4.2). Let $m-1\leqslant\omega<m$. We have

$$\sum_{3\leqslant n\leqslant\omega} n^{-1}(\log_2 n)^{\varepsilon-1}\exp[(\log n)^{\Delta}] \leqslant (\log_2\omega)^{\varepsilon-1}\sum_{3\leqslant n\leqslant\omega} n^{-1}\exp[(\log n)^{\Delta}]$$

and

$$\sum_{3\leqslant n\leqslant \omega} n^{-1}\exp[(\log n)^{\Delta}] < \int_3^{\omega} x^{-1}\exp[(\log x)^{\Delta}]dx + m^{-1}\exp[(\log m)^{\Delta}]$$

$$= \int_3^{\omega} x^{-1}\exp[(\log x)^{\Delta}](\log x)^{\Delta-1}\frac{1}{(\log x)^{\Delta-1}}dx + m^{-1}\exp[(\log m)^{\Delta}]$$

$$= O\left[\frac{\exp[(\log x)^{\Delta}]}{(\log x)^{\Delta-1}}\right]_3^{\omega} + O\left\{\int_3^{\omega}\frac{\exp[(\log x)^{\Delta}]}{x(\log x)^{\Delta}}dx\right\} + O\left\{\omega^{-1}\exp[(\log \omega)^{\Delta}]\right\}$$

$$= O\left\{\frac{\exp[(\log \omega)^{\Delta}]}{(\log \omega)^{\Delta-1}}\right\}.$$

Hence

$$\sum_{3\leqslant n\leqslant \omega} n^{-1}\exp[(\log n)^{\Delta}](\log_2 n)^{\varepsilon-1} = O\left\{\frac{\exp[(\log \omega)^{\Delta}](\log_2 \omega)^{\varepsilon-1}}{(\log \omega)^{\Delta-1}}\right\}.$$

Proof of (1.4.3). By Abel's lemma, we have, for $m-1\leqslant \omega < m$,

$$\eta(\omega, t) = \sum_{3\leqslant n\leqslant \omega}\exp[(\log n)^{\Delta}](\log_2 n)^{\varepsilon-1}n^{-1}\sin nt$$

$$= O\{\exp[(\log \omega)^{\Delta}](\log_2 \omega)^{\varepsilon-1}\omega^{-1}t^{-1}\}.$$

Proof of (1.4.4). We have, by (1.4.1),

$$g(\omega, t) = \int_0^t \big(\log_2(k/u)\big)^{-\varepsilon}\theta(\omega, u)du$$

$$= O\left\{\int_0^t \big(\log_2(k/u)\big)^{-\varepsilon}\frac{\omega\exp[(\log \omega)^{\Delta}](\log_2 \omega)^{\varepsilon-1}}{(\log \omega)^{\Delta-1}}du\right\}$$

$$= O\left\{\frac{\omega t\exp[(\log \omega)^{\Delta}](\log_2 \omega)^{\varepsilon-1}\big(\log_2(k/t)\big)^{-\varepsilon}}{(\log \omega)^{\Delta-1}}\right\}.$$

Proof of (1.4.5). Applying second mean-value theorem, we have, by (1.4.2),

$$g(\omega, t) = \big(\log_2(k/t)\big)^{-\varepsilon}\int_{\zeta}^{t}\theta(\omega, u)du \qquad (0\leqslant \zeta\leqslant t)$$

$$= O\left\{\big(\log_2(k/t)\big)^{-\varepsilon}\sum_{3\leqslant n\leqslant \omega}\exp[(\log n)^{\Delta}](\log_2 n)^{\varepsilon-1}n^{-1}\right\}$$

$$= O\left\{\frac{\exp[(\log \omega)^{\Delta}](\log_2 \omega)^{\varepsilon-1}\big(\log_2(k/t)\big)^{-\varepsilon}}{(\log \omega)^{\Delta-1}}\right\}.$$

Proof of (1.4.6). Using the second mean-value theorem, we have, by (1.4.3),

$$h(\omega, t) = \left(\log_2(k/\pi)\right)^{-\varepsilon} \int_{\xi}^{\pi} \theta(\omega, u)\,du \qquad (t \leqslant \xi \leqslant \pi)$$
$$= -\left(\log_2(k/\pi)\right)^{-\varepsilon} \eta(\omega, \xi)$$
$$= O\{\exp[(\log\omega)^{\Delta}](\log_2\omega)^{\varepsilon-1}\omega^{-1}t^{-1}\}.$$

1.5. For the proof of our theorem, we require the following lemmas:

LEMMA 1. *If the Fourier series of the even function* $(\log_2|k/t|)^{-\varepsilon}$, $\varepsilon \geqslant 1$, *defined outside* $(-\pi, \pi)$ *by periodicity, be* $\sum a_n \cos nt$, *then*

$$\sum |a_n| (\log_2 n)^{\varepsilon-1} < \infty.$$

Proof. The technique of proof is due originally to Mohanty [3]. Let

$$\left(\log_2|(k/t)|\right)^{-\varepsilon} \sim \sum a_n \cos nt.$$

We have

$$a_n = \frac{2}{\pi} \int_0^{\pi} \left(\log_2(k/t)\right)^{-\varepsilon} \cos nt\,dt \qquad (n \geqslant 1).$$

Integrating by parts, we obtain

$$a_n = -\frac{2\varepsilon}{n\pi} \int_0^{\pi} \left(\log_2(k/t)\right)^{-\varepsilon-1} \left(\log(k/t)\right)^{-1} \frac{\sin nt}{t}\,dt$$
$$= -\frac{2\varepsilon}{n\pi}\left(\int_0^{1/n} + \int_{1/n}^{\pi}\right)$$
$$= -\frac{2\varepsilon}{n\pi}(I_1 + I_2).$$

Now, since $|\sin nt| \leqslant nt$, we have

$$I_1 = \int_0^{1/n} \left(\log_2(k/t)\right)^{-\varepsilon-1} \left(\log(k/t)\right)^{-1} \frac{\sin nt}{t}\,dt$$
$$= O\left\{ (\log_2 n)^{-\varepsilon-1} (\log n)^{-1} \int_0^{1/n} \frac{|\sin nt|}{t}\,dt \right\}$$
$$= O\{(\log_2 n)^{-\varepsilon-1}(\log n)^{-1}\}$$

and

$$I_2 = \int_{1/n}^{\pi} \big(\log_2(k/t)\big)^{-\varepsilon-1}\big(\log(k/t)\big)^{-1}\frac{\sin nt}{t}\,dt$$

$$= O\left\{n(\log_2 n)^{-\varepsilon-1}(\log n)^{-1}\left|\int_{1/n}^{\eta}\sin nt\,dt\right|\right\} \quad \left(\frac{1}{n}\leqslant\eta\leqslant\pi\right)$$

$$= O\{(\log_2 n)^{-\varepsilon-1}(\log n)^{-1}\}.$$

Thus

$$|a_n| = O\{n^{-1}(\log_2 n)^{-\varepsilon-1}(\log n)^{-1}\}.$$

Hence the result of the lemma.

LEMMA 2. *The integral*

$$\frac{2}{\pi}\int_{A}^{\infty} \Delta\omega^{-1}(\log\omega)^{\Delta-1}\exp[-(\log\omega)^{\Delta}]\left|\sum_{3\leqslant n\leqslant\omega}\exp[(\log n)^{\Delta}](\log_2 n)^{\varepsilon-1}a_n\right|d\omega$$

is convergent, or, what is the same thing, $\sum a_n(\log_2 n)^{\varepsilon-1}$, *where* a_n *is defined as in Lemma* 1, *is summable* $|R, \exp[(\log\omega)^{\Delta}], 1|$.

Proof. The series $\sum a_n(\log_2 n)^{\varepsilon-1}$, as can be seen by Lemma 1, is absolutely convergent. Hence, by the first theorem of consistency for absolute Riesz summability, the result follows.

1.6. Proof of the theorem. We have

$$A_n(x) = \frac{2}{\pi}\int_0^{\pi}\varphi(t)\cos nt\,dt$$

$$= \frac{2}{\pi}\left[\varphi(t)\big(\log_2(k/t)\big)^{\varepsilon}\int_0^t\big(\log_2(k/u)\big)^{-\varepsilon}\cos nu\,du\right]_0^{\pi} -$$

$$-\frac{2}{\pi}\int_0^{\pi} d\{\varphi(t)\big(\log_2(k/t)\big)^{\varepsilon}\}\int_0^t\big(\log_2(k/u)\big)^{-\varepsilon}\cos nu\,du$$

$$= \varphi(\pi)\big(\log_2(k/\pi)^{\varepsilon}a_n - \frac{2}{\pi}\int_0^{\pi} d\{\varphi(t)\big(\log_2(k/t)\big)^{\varepsilon}\}\times$$

$$\times\int_0^t\big(\log_2(k/u)\big)^{-\varepsilon}\cos nu\,du.$$

The series $\sum_3^{\infty}(\log_2 n)^{\varepsilon-1}A_n(x)$ is summable $|R, \exp[(\log\omega)^{\Delta}], 1|$ if

$$I = \int_A^{\infty}\Delta\omega^{-1}(\log\omega)^{\Delta-1}\exp[-(\log\omega)^{\Delta}]\times$$

$$\times\left|\sum_{3\leqslant n\leqslant\omega}\exp[(\log n)^{\Delta}](\log_2 n)^{\varepsilon-1}A_n(x)\right|d\omega < \infty.$$

Substituting the value of $A_n(x)$ as obtained above, we have

$$I \leqslant \left|\varphi(\pi)\left(\log_2(k/\pi)\right)^{\varepsilon}\right| \int\limits_{A}^{\infty} \Delta\omega^{-1}(\log\omega)^{\Delta-1}\exp[-(\log\omega)^{\Delta}] \times$$
$$\times \left| \sum_{3\leqslant n\leqslant\omega} \exp[(\log n)^{\Delta}](\log_2 n)^{\varepsilon-1} a_n \right| d\omega +$$
$$+ \int\limits_{0}^{\pi} \left| d\{\varphi(t)\left(\log_2(k/t)\right)^{\varepsilon}\}\right| \int\limits_{A}^{\infty} \Delta\omega^{-1}(\log\omega)^{\Delta-1}\exp[-(\log\omega)^{\Delta}] g(\omega, t)\, d\omega .$$

In view of Lemma 2, the first term on the right-hand side of this inequality is finite. And since, by hypothesis,

$$\int\limits_{0}^{\pi} \left| d\{\varphi(t)\left(\log_2(k/t)\right)^{\varepsilon}\}\right| < \infty,$$

it is enough to show that

$$J = \int\limits_{A}^{\infty} \omega^{-1}(\log\omega)^{\Delta-1}\exp[-(\log\omega)^{\Delta}]\, |g(\omega, t)|\, d\omega = O(1),$$

uniformly for $0 < t < \pi$.

Writing $\tau = \left(\log(k/t)\right)^{\varepsilon+\Delta-2}$, we have

$$J = \int\limits_{A}^{\infty} = \int\limits_{A}^{k/t} + \int\limits_{k/t}^{(k/t)\tau} + \int\limits_{(k/t)\tau}^{\infty} = J_1 + J_2 + J_3 .$$

Now, by (1.4.4), we obtain

$$J_1 = O\left\{ \int\limits_{A}^{k/t} \omega^{-1}(\log\omega)^{\Delta-1}\exp[-(\log\omega)^{\Delta}] \times \right.$$
$$\left. \times \frac{\omega t \exp[(\log\omega)^{\Delta}](\log_2\omega)^{\varepsilon-1}\left(\log_2(k/t)\right)^{-\varepsilon}}{(\log\omega)^{\Delta-1}}\, d\omega \right\}$$
$$= O\left\{ \left(\log_2(k/t)\right)^{\varepsilon-1}\left(\log_2(k/t)\right)^{-\varepsilon} t \int\limits_{A}^{k/t} d\omega \right\}$$
$$= O\left\{ \left(\log_2(k/t)\right)^{-1} \right\} = O(1).$$

Using (1.4.5), we have

$$J_2 = O\left\{ \int\limits_{k/t}^{(k/t)\tau} \omega^{-1}(\log\omega)^{\Delta-1}\exp[-(\log\omega)^{\Delta}] \times \right.$$
$$\left. \times \frac{\exp[(\log\omega)^{\Delta}](\log_2\omega)^{\varepsilon-1}\left(\log_2(k/t)\right)^{-\varepsilon}}{(\log\omega)^{\Delta-1}}\, d\omega \right\}$$

$$= O\left\{(\log_2(k/t))^{-\varepsilon}\int_{k/t}^{(k/t)\tau}(\log_2\omega)^{\varepsilon-1}\omega^{-1}d\omega\right\}$$

$$< O\left[(\log_2(k/t))^{-\varepsilon}\{\log_2((k/t)\tau)\}^{\varepsilon-1}\int_{k/t}^{(k/t)\tau}\omega^{-1}d\omega\right]$$

$$= O\left[(\log_2(k/t))^{-\varepsilon}\{\log_2((k/t)\tau)\}^{\varepsilon-1}\log\tau\right]$$

$$= O\left[(\log_2(k/t))^{1-\varepsilon}\{\log_2((k/t)\tau)\}^{\varepsilon-1}\right] = O(1).$$

We now proceed to show that $J_3 = O(1)$.

We have $|g(\omega, t)| \leqslant |g(\omega, \pi)| + |h(\omega, t)|$.

Therefore

$$J_3 \leqslant \int_{(k/t)\tau}^{\infty}\omega^{-1}(\log\omega)^{\Delta-1}\exp[-(\log\omega)^{\Delta}]\,|g(\omega,\pi)|\,d\omega + \\ + \int_{(k/t)\tau}^{\infty}\omega^{-1}(\log\omega)^{\Delta-1}\exp[-(\log\omega)^{\Delta}]\,|h(\omega,t)|\,d\omega.$$

Now, by virtue of Lemma 2, the first term on the right-hand side of this inequality is evidently $O(1)$.

By (1.4.6), we have

$$J_3 \leqslant O(1) + \\ + O\left\{\int_{(k/t)\tau}^{\infty}\omega^{-1}(\log\omega)^{\Delta-1}\exp[-(\log\omega)^{\Delta}]\exp[(\log w)^{\Delta}](\log_2 w)^{\varepsilon-1}w^{-1}t^{-1}\right\}$$

$$= O(1) + O\left\{t^{-1}\int_{(k/t)\tau}^{\infty}\omega^{-2}(\log\omega)^{\Delta-1}(\log_2\omega)^{\varepsilon-1}d\omega\right\}$$

$$= O(1) + O\left\{t^{-1}\int_{(k/t)\tau}^{\infty}\frac{(\log\omega)^{\Delta-1}(\log_2\omega)^{\varepsilon-1}}{\omega^{\varepsilon'}}\,\frac{1}{\omega^{2-\varepsilon'}}\,d\omega\right\}$$

$$= O(1) + O\left[\frac{t^{-1}\{\log((k/t)\tau)\}^{\Delta-1}}{\{(k/t)\tau\}^{\varepsilon'}}\,\frac{\{\log_2((k/t)\tau)\}^{\varepsilon-1}}{\{(k/t)\tau\}^{1-\varepsilon'}}\right]$$

$$= O(1) + O\left[\frac{\{\log((k/t)\tau)\}^{\Delta-1}\{\log_2((k/t)\tau)\}^{\varepsilon-1}}{\tau}\right]$$

$$= O(1) + O(1) = O(1),$$

where ε' is arbitrary, and such that $0 < \varepsilon' < 1$.

Thus the theorem has been completely proved.

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