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ON AXIAL MAPS OF DIRECT PRODUCTS, I

BY

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In this paper we give proofs of the results announced in [2].

1. THEOREMS AND LEMMAS

A function $f\colon A_1\times \ldots \times A_n \to A_1\times \ldots \times A_n$ is called *axial* if there exist i and $g\colon A_1\times \ldots \times A_n \to A_i$ such that

$$f(x_1, \ldots, x_n) = \big(x_1, \ldots, x_{i-1}, g(x_1, \ldots, x_n), x_{i+1}, \ldots, x_n\big)$$

for all $(x_1, \ldots, x_n)\in A_1\times \ldots \times A_n$.

A function $f\colon X\to X$ which is one-to-one and onto is called a *permutation* of X.

Nosarzewska [5] has proved the following theorem:

(0) *If* $|A|\leqslant |B| = \aleph_0$, *then every permutation of* $A\times B$ *can be represented as a composition of* 5 *axial permutations of* $A\times B$.

In this note we shall extend this theorem to A and B of arbitrary finite or infinite cardinality and we shall consider some problems concerning refinements of Proposition 4 of [3].

We shall prove the following results:

(i) *If* $|A| = |B|\geqslant\aleph_0$, *then every permutation* p *of* $A\times B$ *can be represented as a composition*

$$p = p_1\circ \ldots \circ p_5,$$

where all p_i $(i = 1, 2, \ldots, 5)$ *are axial permutations of* $A\times B$ *and* $p_1(a, b) = \big(g(a, b), b\big)$ *for all* $(a, b)\in A\times B$.

(ii) *If* $\aleph_0\leqslant |A| \neq |B|\geqslant\aleph_0$, *then every permutation* p *of* $A\times B$ *can be represented as a composition*

$$p = p_1\circ \ldots \circ p_4,$$

where all p_i $(i = 1, 2, 3, 4)$ *are axial permutations of* $A \times B$ *and* $p_1(a, b) = (g(a, b), b)$ *for all* $(a, b) \in A \times B$.

(iii) *If* $|A| < \aleph_0$ *(while* B *may be of arbitrary finite or infinite cardinality), then every permutation* p *of* $A \times B$ *can be represented as a composition*

$$p = p_1 \circ p_2 \circ p_3,$$

where all p_i $(i = 1, 2, 3)$ *are axial permutations of* $A \times B$ *and* $p_1(a, b) = (g(a, b), b)$ *for all* $(a, b) \in A \times B$.

(iv) *If* A *and* B *are finite, then every function* $f\colon A \times B \to A \times B$ *can be represented as a composition*

$$f = f_1 \circ \ldots \circ f_6,$$

where f_1, f_2, f_5 *and* f_6 *are axial permutations, and* f_3 *and* f_4 *are axial functions.*

In connection with (i) let us mention that Eggleston has studied in [1] axial homeomorphisms of the plane $R \times R$.

The following problem is open:

Can one decrease the number 6 in (iv)? **(P 910)**

We can prove only the following

(v) *If* $|A| = |B| \geqslant \aleph_0$, *then there exists a permutation* r *of* $A \times B$ *such that, for any axial permutations* r_1, r_2, r_3, r_4 *of* $A \times B$, *we have* $r \neq r_1 \circ \ldots \circ r_4$ *whenever* r_1 *is of the form* $r_1(a, b) = (g(a, b), b)$ *for all* $(a, b) \in A \times B$.

(v′) *If* A *and* B *are infinite, then there exists a permutation* p *of* $A \times B$ *such that, for any axial permutations* p_1, p_2, p_3 *of* $A \times B$, *we have* $p \neq p_1 \circ p_2 \circ p_3$.

(v″) *If* $|A| \geqslant 2$ *and* $|B| \geqslant 2$, *then there exists a permutation* p *of* $A \times B$ *such that, for any axial permutations* p_1, p_2 *of* $A \times B$, *we have* $p \neq p_1 \circ p_2$.

We can partially extend (i)-(iv) as follows:

(vi) *If* $A_1, \ldots, A_m$ *are finite and* $B_1, \ldots, B_n$ *are infinite, then every permutation* p *of* $A_1 \times \ldots \times A_m \times B_1 \times \ldots \times B_n$ *can be represented as a composition*

$$p = p_1 \circ \ldots \circ p_{k(m,n)},$$

where all p_i $(i = 1, 2, \ldots, k(m, n))$ *are axial permutations and*

$$p_1(x_1, \ldots, x_{m+n}) = (g(x_1, \ldots, x_{m+n}), x_2, \ldots, x_{m+n})$$

for every $(x_1, \ldots, x_{m+n}) \in A_1 \times \ldots \times A_m \times B_1 \times \ldots \times B_n$, $k(m, 0) = 2m-1$ *and* $k(m, n) = 2m + l(n)$ *for* $n \geqslant 1$, *with* $l(1) = 1$ *and, for* $n > 1$, $l(n) = \min\{2l(r) + 3l(s)\colon r$ *and* s *are integers less than* n *and* $r + s = n\}$.

(vii) *If at least one of the sets $A_1, \ldots, A_n$ is infinite, then every function $f\colon A_1 \times \ldots \times A_n \to A_1 \times \ldots \times A_n$ can be represented as a composition*

$$f = f_1 \circ \ldots \circ f_{n+1},$$

where all f_i $(i = 1, 2, \ldots, n+1)$ are axial functions.

The following fact, obvious for finite sets, will be useful:

(viii) *If at least one of the sets $A_1, \ldots, A_n$ is infinite, then every function $f\colon A_1 \times \ldots \times A_n \to A_1 \times \ldots \times A_n$ which is onto can be represented as a composition*

$$f = f_1 \circ p,$$

where f_1 is an axial function onto and p is a permutation of $A_1 \times \ldots \times A_n$.

If, moreover, $|A_1| \geqslant |A_i|$ for each $2 \leqslant i \leqslant n$, then f_1 can be expressed by the formula

$$f_1(x_1, \ldots, x_n) = \big(g(x_1, \ldots, x_n), x_2, \ldots, x_n\big) \quad \text{for all } (x_1, \ldots, x_n) \in A_1 \times \ldots \times A_n.$$

It follows easily from (vi) and (viii) that

(ix) *If $A_1, \ldots, A_m$ are finite and $B_1, \ldots, B_n$ are infinite, then every function*

$$f\colon A_1 \times \ldots \times A_m \times B_1 \times \ldots \times B_n \to A_1 \times \ldots \times A_m \times B_1 \times \ldots \times B_n$$

which is onto can be represented as a composition

$$f = f_1 \circ \ldots \circ f_{t(m,n)},$$

where f_1 is an axial function onto, $f_2, \ldots, f_{t(m,n)}$ are axial permutations, and

$$t(m, n) = \begin{cases} k(m, n) & \text{if } m = 0 \text{ or } n = 0, \\ k(m, n) + 1 & \text{if } m > 0 \text{ and } n > 0 \end{cases}$$

with $k(m, n)$ defined as in (vi).

To prove our results we need some lemmas.

The following fact is known (see [4], Theorem 10.1.5):

(x) *If a is a positive integer and $\boldsymbol{P}$ and $\boldsymbol{Q}$ are two partitions of a set X into a-element sets, then there exists a set $S \subseteq X$ such that $|Y \cap S| = 1$ for every $Y \in \boldsymbol{P} \cup \boldsymbol{Q}$.*

From (x) we get immediately the following lemma:

(xi) *If $a, \boldsymbol{P}, \boldsymbol{Q}$ and X are as in* (x), *then there exists a partition $\boldsymbol{R}$ of X into sets such that $|Y \cap R| = 1$ for every $R \in \boldsymbol{R}$ and every $Y \in \boldsymbol{P} \cup \boldsymbol{Q}$.*

In the next section we give proofs of the following Lemmas (xii) and (xiii).

(xii) *If $\boldsymbol{P}$ and $\boldsymbol{Q}$ are two partitions of a set X into sets such that $|\boldsymbol{P}| = |\boldsymbol{Q}|$ and $|P_1 \cap Q_1| = |P_2 \cap Q_2|$ for every $P_1, P_2 \in \boldsymbol{P}$ and every $Q_1, Q_2 \in \boldsymbol{Q}$,* *then there exists a partition* $\boldsymbol{R}$ *of* X *into sets such that* $|Y\cap R| = 1$ *for every* $Y\in \boldsymbol{P}\cup \boldsymbol{Q}$ *and every* $R\in \boldsymbol{R}$.

Nosarzewska has implicite proved in [5] Lemma (xii) in the countable case.

(xiii) *If* p *and* q *are permutations of* $A\times B$, *and* $\boldsymbol{P}, \boldsymbol{Q}$ *are the partitions of* $X = A\times B$,

$$\boldsymbol{P} = \{p(A\times\{b\}):\ b\in B\},\qquad \boldsymbol{Q} = \{q(A\times\{b\}):\ b\in B\},$$

then the following two conditions are equivalent:

(a) *There exist axial permutations* r_1, r_2, r_3 *of* X *such that*

$$p = q\circ r_1\circ r_2\circ r_3 \quad \text{and} \quad r_1(a, b) = \big(g(a, b), b\big) \quad \text{for all } (a, b)\in X.$$

(b) *There exists a partition* $\boldsymbol{R}$ *of* X *into sets such that* $|Y\cap R| = 1$ *for every* $R\in\boldsymbol{R}$ *and every* $Y\in\boldsymbol{P}\cup\boldsymbol{Q}$.

Obviously, we can write (xiii) also as follows:

(xiii′) *If* p *and* q *are permutations of* $A\times B$, *and* $\boldsymbol{P}', \boldsymbol{Q}'$ *are the partitions of* $X = A\times B$,

$$\boldsymbol{P}' = \{p(\{a\}\times B):\ a\in A\},\qquad \boldsymbol{Q}' = \{q(\{a\}\times B):\ a\in A\},$$

then the following two conditions are equivalent:

(a′) *There exist axial permutations* r_1, r_2, r_3 *of* X *such that*

$$p = q\circ r_1\circ r_2\circ r_3 \quad \text{and} \quad r_1(a, b) = \big(a, g(a, b)\big) \quad \text{for all } (a, b)\in X.$$

(b′) *There exists a partition* $\boldsymbol{R}$ *of* X *into sets such that* $|Y\cap R| = 1$ *for every* $R\in\boldsymbol{R}$ *and every* $Y\in\boldsymbol{P}'\cup\boldsymbol{Q}'$.

Let us note that using the rather simple Lemma (xiii) it is not hard to see that (xi) (or (x)) and (iii) are equivalent (i.e., each one follows easily from the other).

2. *PROOFS*

A matrix $D = (d_{a,b})_{a\in A, b\in B}$ is called a *permutation of a matrix* $C = (c_{a,b})_{a\in A, b\in B}$ if there exists a permutation $r = (r', r'')$ of $A\times B$ such that $d_{a,b} = c_{r'(a,b), r''(a,b)}$ for all $a\in A$, $b\in B$.

In the sequel we shall use the following definition:

Let A and B be sets, let $X = A\times B$, and let M be the matrix $\big((a, b)\big)_{a\in A, b\in B}$. For every permutation q of X, we put $q(M) = \big(q(a, b)\big)_{a\in A, b\in B}$. Notice that, for any vertical (horizontal) permutation N of the matrix $q(M)$, i.e. a permutation such that the set of elements of each column (row) of the matrix N coincides with the set of elements of the column (row) of $q(M)$ with the same index, there exists an axial permutation r of X such that $q\circ r(M) = N$ and $r(a, b) = (g(a, b), b)$ for all $(a, b)\in X$ ($r(a, b) = (a, g(a, b))$ for all $(a, b)\in X$). Conversely, for any axial permutation r of X such that $r(a, b) = (g(a, b), b)$ for all $(a, b)\in X$ ($r(a, b) = (a, g(a, b))$ for all $(a, b)\in X$), the matrix $q\circ r(M)$ is a vertical (horizontal) permutation of the matrix $q(M)$.

First, we prove Lemmas (xii) and (xiii).

Proof of (xii).

SUBLEMMA. *If sets Y and Z are of the same cardinality, then there exists a family F of one-to-one functions from Y onto Z such that*

$$Y\times Z = \bigcup_{f\in F} \operatorname{graph}(f) \quad \textit{and} \quad \operatorname{graph}(f_1)\cap\operatorname{graph}(f_2) = \emptyset$$
$$\textit{for all } f_1 \neq f_2 \ (f_1, f_2\in F).$$

Indeed, we can assume, without loss of generality, that $Y = Z \neq \emptyset$. It is well known that, for every cardinal number greater than 0, there exists a group of this cardinality. Hence we can assume that Y is equipped with the group structure. Let $F = \{f_t\colon t\in Y\}$, where $f_t(y) = t\cdot y$ for every $t, y\in Y$. (Here $\cdot$ is the group operation in Y.) It is easy to check that F is what we need.

Let $\boldsymbol{P}$, $\boldsymbol{Q}$ and X be as in (xii). Put $Y = \boldsymbol{P}$, $Z = \boldsymbol{Q}$, and let F be a family of functions with the properties listed in the Sublemma. For every $f\in F$, put

$$X_f = \bigcup_{P\in \boldsymbol{P}} P\cap f(P).$$

The family $\{P\cap f(P)\}_{P\in\boldsymbol{P}}$ is a partition of X_f. From the assumption of (xii) we have $|P_1\cap f(P_1)| = |P_2\cap f(P_2)|$ for all $f\in F$ and for all $P_1, P_2\in\boldsymbol{P}$. Hence, for any $f\in F$, we can define a partition $\boldsymbol{R}_f$ of X_f such that every $R\in\boldsymbol{R}_f$ is a selector of the family $\{P\cap f(P)\}_{P\in\boldsymbol{P}}$. By the property of F we infer that the family $\{X_f\}_{f\in F}$ is a partition of X. Put

$$\boldsymbol{R} = \bigcup_{f\in F} \boldsymbol{R}_f.$$

It is easy to see that $\boldsymbol{R}$ is a desired partition of X.

Proof of (xiii). Let p, q, A, B, X and $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}$ be as in (xiii).

(b) $\Rightarrow$ (a). Let M be the matrix $((a, b))_{a\in A, b\in B}$. Then $\boldsymbol{P}$ and $\boldsymbol{Q}$ are the sets of columns of $p(M)$ and $q(M)$, respectively. Let $\boldsymbol{R}$ be a partition of X as in (b). Thus each set $R\in\boldsymbol{R}$ has exactly one element in common with each column of $p(M)$ and one element in common with each column of $q(M)$. Therefore, there exists a vertical permutation of $q(M)$, and hence an axial permutation r_1 of X, such that the sets $R\in\boldsymbol{R}$ are the rows of $q\circ r_1(M)$. Now, there exists a horizontal permutation of $q\circ r_1(M)$, and hence an axial permutation r_2 of X, such that the set of elements of each column of $q\circ r_1\circ r_2(M)$ coincides with the set of elements of the column of $p(M)$ with the same index. Thus there exists a vertical permutation of $q\circ r_1\circ r_2(M)$, and hence an axial permutation r_3 of X, such that $q\circ r_1\circ r_2\circ r_3(M) = p(M)$. Hence

$$p = q\circ r_1\circ r_2\circ r_3.$$

The additional claim on r_1 is also visible from the definition of r_1.

(a) $\Rightarrow$ (b). Let r_1, r_2, r_3 be as in (a). Hence

$$r_1(a, b) = \big(g_1(a, b), b\big) \quad \text{for all } (a, b)\in X.$$

Obviously, we can additionally assume, without loss of generality, that

$$r_2(a, b) = \big(a, g_2(a, b)\big) \quad \text{for all } (a, b)\in X$$

and

$$r_3(a, b) = \big(g_3(a, b), b\big) \quad \text{for all } (a, b)\in X.$$

Let M be the matrix $\big((a, b)\big)_{a\in A, b\in B}$. For every $a\in A$, we put $R_a = q\circ r_1(\{a\}\times B)$. Take $\boldsymbol{R} = \{R_a\colon a\in A\}$. Thus $\boldsymbol{R}$ is the set of rows of $q\circ r_1(M)$. We prove that the partition $\boldsymbol{R}$ of X satisfies (b). The set of elements of each column of $q\circ r_1(M)$ coincides with the set of elements of the column of $q(M)$ with the same index. Thus $|R\cap Q| = 1$ for every $Q\in \boldsymbol{Q}$ and every $R\in \boldsymbol{R}$.

Since $\boldsymbol{R}$ is the family of rows of $q\circ r_1(M)$, each $R\in\boldsymbol{R}$ has exactly one element in common with each column of $q\circ r_1\circ r_2(M)$. Since columns in $q\circ r_1\circ r_2(M)$ and $q\circ r_1\circ r_2\circ r_3(M)$ have the same sets of elements if they have the same index, each $R\in\boldsymbol{R}$ has exactly one element in common with each column of $q\circ r_1\circ r_2\circ r_3(M)$. Hence $|P\cap R| = 1$ for every $P\in\boldsymbol{P}$ and every $R\in\boldsymbol{R}$.

Proof of (i). The idea of our proof of (i) is the same as that of the proof of (0) given by Nosarzewska in [5]. Theorem (i) follows from the following

THEOREM (0*). *If $|A| \leqslant |B| \geqslant \aleph_0$, then every permutation p of $A\times B$ can be represented as a composition*

$$p = p_1\circ \ldots \circ p_5,$$

where all p_i $(i = 1, 2, \ldots, 5)$ are axial permutations of $A\times B$ and $p_1(a, b) = \big(a, g_1(a, b)\big)$ for all $(a, b)\in A\times B$.

Proof. Let $X = A\times B$ and $M = \big((a, b)\big)_{a\in A, b\in B}$. Assume that X and B are well ordered. Let T be the set of all ordinals of power less than $\mathfrak{m} = |B|$. Let a function $f\colon T\to A$ be such that $|f^{-1}(a)| = \mathfrak{m}$ for all $a\in A$.

First step. We define a permutation p_1 of X of the form $p_1(a, b) = (a, g_1(a, b))$ for all $(a, b) \in X$, having the following property:

$(*)$ There exists a transfinite sequence $\{x_t\}_{t\in T}$ of elements of X such that each column of $p_1(M)$ has at most one element appearing in this sequence, and each row of $p(M)$ has $\mathfrak{m}$ elements of this sequence.

First, we define, by transfinite induction, a sequence $\{(a_t, b_t)\}_{t\in T}$ of elements of X and a sequence $\{c_t\}_{t\in T}$ of elements of B with the property

$(**)$ $c_t \neq c_{t'}$ whenever $t \neq t'$ $(t, t' \in T)$ and, for every $t \in T$, (a_t, b_t) appears in the $f(t)$-th row of $p(M)$,

$$(a_t, b_t) \notin \{(a_{t'}, b_{t'})\colon t' < t\} \cup \{(a_{t'}, c_{t'})\colon t' < t\},$$

and

$$c_t \notin \{b_{t'}\colon t' \leqslant t\} \cup \{c_{t'}\colon t' < t\}.$$

Let (a_0, b_0) be the smallest element of X appearing in the $f(0)$-th row of $p(M)$, and let c_0 be the smallest element of $B \setminus \{b_0\}$. Assume that the elements $(a_{t'}, b_{t'})$ and $c_{t'}$ for $t' < t$ have been already defined.

Let (a_t, b_t) be the smallest element of

$$X \setminus \big(\{(a_{t'}, b_{t'})\colon t' < t\} \cup \{(a_{t'}, c_{t'})\colon t' < t\}\big)$$

which appears in the $f(t)$-th row of $p(M)$, and let c_t be the smallest element of

$$B \setminus (\{b_{t'}\colon t' \leqslant t\} \cup \{c_{t'}\colon t' < t\}).$$

The just defined two sequences have property $(**)$, which implies

$(***)$ $(a_t, b_t) \neq (a_{t'}, c_{t'})$ for $t, t' \in T$, and $(a_t, b_t) \neq (a_{t'}, b_{t'})$ and $(a_t, c_t) \neq (a_{t'}, c_{t'})$ for $t \neq t'$ $(t, t' \in T)$.

Now we define a permutation p_1 as follows:

$$p_1(a, b) = \begin{cases} (a, b) & \text{if } (a, b) \notin \{(a_t, b_t)\colon t \in T\} \cup \{(a_t, c_t)\colon t \in T\}, \\ (a_t, b_t) & \text{if } (a, b) = (a_t, c_t), \\ (a_t, c_t) & \text{if } (a, b) = (a_t, b_t). \end{cases}$$

Property $(***)$ shows that the definition of p_1 is correct, and $(**)$ implies that p_1 has property $(*)$ (we put $x_t = p_1(a_t, c_t)$ for $t \in T$).

Second step. Since the matrix $p_1(M)$ has property $(*)$, there exists a vertical permutation of $p_1(M)$, and hence an axial permutation p_2 of X, such that each row of $p_1 \circ p_2(M)$ has $\mathfrak{m}$ elements in common with each row of $p(M)$.

Third step. Put $q = p_1 \circ p_2$. From the property of q (see second step) and Lemma (xii) it follows that condition (b′) of Lemma (xiii′) is satisfied for p and q, and so does condition (a′) of this lemma. Hence

$p = q \circ r_1 \circ r_2 \circ r_3$. Put $p_3 = r_1$, $p_4 = r_2$ and $p_5 = r_3$. Now $p = p_1 \circ \ldots \circ p_5$. The additional claim on p_1 is also visible from the definition of p_1.

Remark. If in Theorem (0*) we assume additionally that $|A| < |B|$, then p_1 can be the identity.

Proof. Let $A, B, p, X, \mathfrak{m}, M, T, f$ be as in the proof of Theorem (0*). By the proof of Theorem (0*), it can be seen that to prove our Remark it is sufficient to show that if $|A| < |B| \geqslant \aleph_0$, then there exists an infinite sequence $\{x_t\}_{t \epsilon T}$ of elements of X such that in each column of M appears at most one element of this sequence and each row of $p(M)$ has $\mathfrak{m}$ elements of this sequence. We define, by transfinite induction, this transfinite sequence. Let x_0 be the smallest element of X appearing in the $f(0)$-th row of $p(M)$. Assume that the elements $x_{t'}$ for $t' < t$ have been already defined. Let x_t be the smallest element of X which appears in the $f(t)$-th row of $p(M)$ and such that the column of M containing x_t has not any common element with the set $\{x_{t'}: t' < t\}$. The defined transfinite sequence $\{x_t\}_{t \epsilon T}$ has all desired properties.

Proof of (ii). It follows from Theorem (0*) and the Remark that

(I) If $|A| < |B| \geqslant \aleph_0$, then every permutation p of $A \times B$ can be represented as a composition

$$p = p_1 \circ \ldots \circ p_4,$$

where all p_i $(i = 1, 2, 3, 4)$ are axial permutations of $A \times B$ and

$$p_1(a, b) = (g_1(a, b), b), \quad \ldots, \quad p_4(a, b) = (a, g_4(a, b)) \quad \text{for all } (a, b) \epsilon A \times B.$$

Now we observe that

(II) If $|A| < |B| \geqslant \aleph_0$, then every permutation r of $A \times B$ can be represented as a composition

$$r = r_1 \circ \ldots \circ r_4,$$

where all r_i $(i = 1, 2, 3, 4)$ are axial permutations of $A \times B$ and $r_1(a, b) = (a, g(a, b))$ for all $(a, b) \epsilon A \times B$.

Indeed, put $p = r^{-1}$ in (I). Let p_1, p_2, p_3, p_4 be as in (I). Hence $r^{-1} = p = p_1 \circ \ldots \circ p_4$. Thus $r = p_4^{-1} \circ \ldots \circ p_1^{-1}$. Put $r_1 = p_4^{-1}$, $r_2 = p_3^{-1}$, $r_3 = p_2^{-1}$, $r_4 = p_1^{-1}$. Hence $r = r_1 \circ \ldots \circ r_4$. The additional claim on r_1 can be easily concluded from the definition of r_1.

Theorem (ii) follows almost immediately from (I) and (II).

Proof of (iii). Let A be finite and let p be a permutation of $A \times B$. Let in Lemma (xiii) q be equal to the identity. By Lemma (xi), condition (b) in (xiii) is satisfied, and so does condition (a).

Proof of (iv). Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$ with $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$. Let $X = A \times B$ and $M = \big((a, b)\big)_{a \in A, b \in B}$. We order X as follows (see Fig. 1):

$(a_i, b_j) < (a_k, b_l)$ iff $\{$either $i < k$ or $[i = k$ and $((i$ is odd and $j < l)$ or $(i$ is even and $j > l))]\}$.

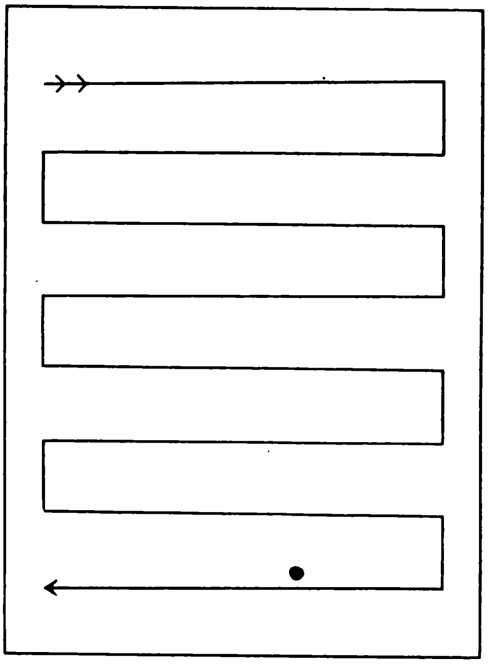

Fig. 1

We partition X into intervals I_{ab} with respect to $<$ such that

$$\operatorname{card}(I_{ab}) = \operatorname{card}\big(f^{-1}(a, b)\big) \quad \text{for all } (a, b) \in f(X).$$

Let J_{ab} be the (or one of the) longest intervals in I_{ab} which is contained in one row of M. Let now p be a permutation of X such that $(a, b) \in p(J_{ab})$ for every $(a, b) \in f(X)$. Hence there exists a horizontal function g over the matrix M such that $p \circ g(J_{ab}) = \{(a, b)\}$ for every $(a, b) \in f(X)$. Therefore, since every column of $p \circ g(M)$ which has elements with indices in I_{ab} has also elements with indices in J_{ab}, there exists a vertical function h over M such that $p \circ g \circ h(I_{ab}) = \{(a, b)\}$ for all $(a, b) \in f(X)$. Now, of course, there exists a permutation q of X such that $p \circ g \circ h \circ q(M) = f(M)$. Now, since both A and B are finite, by (iii) we can represent p as $f_1 \circ f_2 \circ u$ and q as $v \circ f_5 \circ f_6$, where f_1, f_2, u, v, f_5, f_6 are axial permutations, u is a horizontal permutation, and v is a vertical permutation. Thus, putting $f_3 = u \circ g$ and $f_4 = h \circ v$, the functions f_3 and f_4 are axial, and $f_1 \circ \ldots \circ f_6(M) = f(M)$.

Proof of (v). Let $X = A \times B$ and $M = \big((a, b)\big)_{a \in A, b \in B}$. Let r be a permutation of X such that there is a row of $r(M)$ which contains all but one columns of M. Suppose that there exist axial permutations r_1, r_2, r_3, r_4

of X such that $r_1(a, b) = (g_1(a, b), b), \ldots, r_4(a, b) = (a, g_4(a, b))$ for every $(a, b) \in X$, and $r = r_1 \circ \ldots \circ r_4$. Let in Lemma (xiii) q be equal to the identity and $p = r \circ r_4^{-1}$. It is easy to see that condition (b) of (xiii) is not satisfied for those p and q. Hence (a) of (xiii) is not satisfied. But $p = r \circ r_4^{-1} = r_1 \circ r_2 \circ r_3$, a contradiction.

Proof of (v′). Let $X = A \times B$ and $M = ((a, b))_{a \in A, b \in B}$. Let p be a permutation of X such that there are two rows of M contained in a row of $p(M)$, except finitely many elements at most, and there are two columns of M contained in a column of $p(M)$, except finitely many elements at most. Obviously, such a permutation exists. Let in Lemmas (xiii) and (xiii′) q be equal to the identity. It is easy to check that neither (b) in (xiii) nor (b′) in (xiii′) are satisfied. Hence neither (a) in (xiii) nor (a′) in (xiii′) are satisfied.

Proof of (v″). Let $X = A \times B$ and $M = ((a, b))_{a \in A, b \in B}$. Let p be a permutation of X such that there are a row of $p(M)$ and a column of M having at least two common elements, and there are a column of $p(M)$ and a row of M having at least two common elements.

It is easy to see that p cannot be represented as a composition of two axial permutations.

Proof of (vi). From (i) and (ii) we get immediately the following

PROPOSITION. *If A and B are sets of arbitrary infinite cardinality, then every permutation p of $A \times B$ can be represented as a composition*

$$p = p_1 \circ \ldots \circ p_5,$$

where all p_i $(i = 1, 2, \ldots, 5)$ are axial permutations of $A \times B$ and $p_1(a, b) = (g(a, b), b)$ for all $(a, b) \in A \times B$.

If $m = 0$, then (vi) follows, by an obvious induction, from the Proposition. Now, on account of (iii), we can prove (vi) by induction with respect to m.

Proof of (vii). We can assume, without loss of generality, that $\operatorname{card}(A_1) \geqslant \operatorname{card}(A_i)$ for $i = 2, \ldots, n$. Thus A_1 is infinite and there exists a one-to-one function $g\colon A_1 \times \ldots \times A_n \to A_1$.

Let us define an axial function

$$f_{n+1}(a_1, \ldots, a_n) = (g(a_1, \ldots, a_n), a_2, a_3, \ldots, a_n).$$

Hence there exist functions $g_i\colon A_1 \to A_i$ such that

$$f = (g_1 \circ g, \ldots, g_n \circ g).$$

Therefore, we get axial functions

$$f_i(a_1, \ldots, a_n) = (a_1, \ldots, a_{i-1}, g_i(a_1), a_{i+1}, \ldots, a_n) \quad \text{for } i = 1, \ldots, n$$

which satisfy $f = f_1 \circ \ldots \circ f_{n+1}$.

Proof of (viii). Put $X = A_1 \times \ldots \times A_n$. From the assumption of (viii) we infer that there exists an integer i_0, $1 \leqslant i_0 \leqslant n$, such that $|X| = |A_{i_0}| \geqslant \aleph_0$. Hence we can partition X into sets $E_{a_1,\ldots,a_n}$ such that

$$\operatorname{card}(E_{a_1,\ldots,a_n}) = \operatorname{card}\big(f^{-1}(a_1, \ldots, a_n)\big)$$

and

$$E_{a_1,\ldots,a_n} \subset \{a_1\} \times \ldots \times \{a_{i_0-1}\} \times A_{i_0} \times \{a_{i_0+1}\} \times \ldots \times \{a_n\} \quad \text{for all } (a_1, \ldots, a_n) \in X.$$

Put $f_1(x) = (a_1, \ldots, a_n)$ if $x \in E_{a_1,\ldots,a_n}$. Clearly, f_1 is a function from X onto X and

$$f_1(a_1, \ldots, a_n) = \big(a_1, \ldots, a_{i_0-1}, g(a_1, \ldots, a_n), a_{i_0+1}, \ldots, a_n\big) \quad \text{for all } (a_1, \ldots, a_n) \in X.$$

Since $|f^{-1}(a_1, \ldots, a_n)| = |E_{a_1,\ldots,a_n}|$, there exists a permutation p of X such that

$$p\big(f^{-1}(a_1, \ldots, a_n)\big) = E_{a_1,\ldots,a_n} \quad \text{for all } (a_1, \ldots, a_n) \in X.$$

From the definition of f_1 and p we have $f = f_1 \circ p$.

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Added in proof. If $|A| = |B| \geqslant \aleph_0$, then also every permutation of $A \times B$ can be represented as a composition of 4 axial permutations of $A \times B$ (cf. Theorem (i)). The proof is in preparation for Colloquium Mathematicum.

REFERENCES

[1] H. G. Eggleston, *A property of plane homeomorphisms*, Fundamenta Mathematicae 42 (1955), p. 61-74.

[2] A. Ehrenfeucht and E. Grzegorek, *On axial maps of direct products*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 22 (1974), p. 225-227.

[3] R. McKenzie, J. Mycielski and D. Thompson, *On Boolean functions and connected sets*, Mathematical Systems Theory 5 (1971), p. 259-270.

[4] L. Mirsky, *Transversal theory*, New York and London 1971.

[5] M. Nosarzewska, *On a Banach's problem on infinite matrices*, Colloquium Mathematicum 2 (1951), p. 194-197.

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