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FINITE-DIMENSIONAL SCHAUDER DECOMPOSITIONS IN CERTAIN FRÉCHET SPACES

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A net $\{S_d\colon d\in D\}$ of continuous linear projections with finite-dimensional range on a linear topological space X is said to be a *Schauder operator basis* — S.O.B. (resp. *Schauder dual operator basis* — S.D.O.B.) for X if for each $x\in X$, the net $\{S_d(x)\colon d\in D\}$ is bounded and converges to x, and $S_dS_e = S_e$ (resp. $S_eS_d = S_e$) whenever $e\leqslant d$.

S.O.B.'s and S.D.O.B.'s are discussed extensively in [2]. The motivation for the above definition is the easily verified fact that a sequence $\{S_n\}_{n=1}^{\infty}$ of linear operators on a linear topological space X is both an S.O.B. and S.D.O.B. for X if and only if $\{S_n\}_{n=1}^{\infty}$ is the sequence of partial sum operators associated with a finite dimensional Schauder decomposition for X (cf. [2], Theorem II.1). In [2], Schauder decompositions are called *Schauder bases of subspaces.*

In [1], Theorem 1, we showed that a separable Banach space which admits an S.D.O.B. must also admit a finite dimensional Schauder decomposition. (In [1], S.D.O.B.'s are called *dual π_λ-decompositions* and S.O.B.'s are called *π_λ-decompositions.*) In this note we generalize this theorem to obtain

THEOREM. *Let X be a separable Fréchet space on which there is a continuous norm. If X admits an S.D.O.B., then X also admits a finite dimensional Schauder decomposition.*

Proof. Since X is separable and metrizable, the proof of Lemma 1 in [1] shows that there is a sequence $\{S_n\}_{n=1}^{\infty}$ which forms an S.D.O.B. for X. Since there is a continuous norm on X, there is a sequence $\{\|\ \|_n\}_{n=1}^{\infty}$ of continuous norms on X which generates the Fréchet-space topology on X and such that for each $x\in X$, $\{\|x\|_n\}_{n=1}^{\infty}$ is a non-decreasing sequence. Let B denote $\{x\in X\colon \|x\|_1\leqslant 1\}$.

We now define a sequence $\{p(n)\}_{n=1}^{\infty}$ of positive integers by induction. Let $p(1)=1$. Suppose $p(1)<p(2)<\ldots<p(n)$ have been defined. Choose a positive integer $p(n+1)>p(n)$ such that

$$\text{(1)}\qquad \text{if } x\in \Big(\operatorname{sp}\bigcup_{i=1}^{n} S_{p(i)}[X]\Big)\cap B, \text{ then } \|x-S_{p(n+1)}(x)\|_n\leqslant \frac{1}{2^{n+1}}.$$

This choice is possible because $\|\ \|_1$ is a norm, so that

$$\Big(\operatorname{sp}\bigcup_{i=1}^{n} S_{p(i)}[X]\Big)\cap B$$

is bounded, closed and finite-dimensional, and thus is compact. $\{S_n\}_{n=1}^{\infty}$, being pointwise convergent to the identity operator I, is equicontinuous by the uniform boundedness principle and thus converges to I uniformly on compact sets.

Note that (1) is equivalent to:

$$\text{(2)}\qquad \text{if } x\in \operatorname{sp}\bigcup_{i=1}^{n} S_{p(i)}[X], \text{ then } \|x-S_{p(n+1)}(x)\|_n \leqslant \frac{1}{2^{n+1}}\|x\|_1.$$

Now for $j\geqslant n$, let $T_n^j = S_{p(j)}S_{p(j-1)}\ldots S_{p(n)}$. We will show that $\{T_n^j\colon n=1,2,\ldots, j=n,n+1,\ldots\}$ is equicontinuous. It is sufficient by the uniform boundedness principle to show that $\{T_n^j\}$ is pointwise bounded.

We assert that if $x\in X$ and $j\geqslant n>m$, then

$$\text{(3)}\qquad \|T_n^j(x)\|_m \leqslant \Big(1+\sum_{i=n}^{j-1}\frac{1}{2^i}\Big)\|S_{p(n)}(x)\|_m.$$

If $j=n$, (3) is obvious. If (3) holds for some fixed j, $j\geqslant n$, then

$$\begin{aligned}\|T_n^{j+1}(x)\|_m &\leqslant \|S_{j+1}T_n^j(x)-T_n^j(x)\|_m+\|T_n^j(x)\|_m\\ &\leqslant \frac{1}{2^{j+1}}\|T_n^j(x)\|_1+\|T_n^j(x)\|_m \leqslant \frac{1}{2^{j+1}}2\|S_{p(n)}(x)\|_m+\|T_n^j(x)\|_m\\ &\leqslant \frac{1}{2^j}\|S_{p(n)}(x)\|_m+\Big(1+\sum_{i=n}^{j-1}\frac{1}{2^i}\Big)\|S_{p(n)}(x)\|_m\\ &= \Big(1+\sum_{i=1}^{j}\frac{1}{2^i}\Big)\|S_{p(n)}(x)\|_m,\end{aligned}$$

where the second inequality follows from (2) and the third and fourth inequalities follow from (3). Thus (3) holds if $j+1$ is substituted for j and hence (3) is verified by induction.

Let $x\in X$. Since $\{S_n\}_{n=1}^{\infty}$ is pointwise bounded, there is for each fixed m a positive number K_m such that

$$\|S_n(y)\|_m\leqslant K_m \quad \text{if } y\in\{x\}\cup\{T_i^j(x)\colon 1\leqslant i\leqslant j\leqslant m+1\} \text{ and } n=1,2,\ldots$$

From this and (3) it follows that if $1\leqslant n\leqslant j$, then $\|T_n^j(x)\|_m\leqslant 2K_m$. Thus $\{T_n^j\colon n=1,2,\ldots, j=n,n+1,\ldots\}$ is pointwise bounded and hence equicontinuous.

As in the proof of Theorem 1 of [1], for each n we can let T_n be the pointwise limit of the sequence $\{T_n^j\}_{j=n}^{\infty}$. $\{T_n\}_{n=1}^{\infty}$ is both an S.O.B. and S.D.O.B. for X, and hence X has a finite dimensional Schauder decomposition. This completes the proof.

We asked in [1] whether a separable Banach space X which admits an S.O.B. must also admit a finite dimensional Schauder decomposition and showed in Theorem 3 of [1] that with one extra condition (which is satisfied, for example, if X admits an S.O.B. $\{S_d\colon d \in D\}$ with $\liminf_d \|S_d\| = 1$) the answer is yes. If X is reflexive, this question is equivalent to asking whether X^* admits a finite dimensional Schauder decomposition. Now if $\{S_d\colon d \in D\}$ is an S.O.B. for X, then $\{S_d^*\colon d \in D\}$ is a weak* S.D.O.B. for X^* and is hence a weak S.D.O.B. for X^* for reflexive X. If $\{S_d^*\colon d \in D\}$ were in fact an S.D.O.B. for X^* when X^* is given the norm topology, we could apply the above theorem (or Theorem 1 of [1]) to conclude that X^* admits a finite dimensional Schauder decomposition. Unfortunately, $\{S_d^*\}$ need not be an S.D.O.B. for X^*.

Example. Define $S_n\colon l_p \to l_p$ $(1 < p < \infty)$ by $S_n(\{x_i\}_{i=1}^{\infty}) = \{z_i\}_{i=1}^{\infty}$, where

$$z_i = \begin{cases} x_i + x_{i+n} & \text{if } i \leqslant n, \\ 0 & \text{if } i > n. \end{cases}$$

Clearly, $\{S_n\}_{n=1}^{\infty}$ is an S.O.B. for $l_p \cdot S_n^*\colon l_q \to l_q$ $(1/p + 1/q = 1)$ is defined by $S_n^*(\{x_i\}_{i=1}^{\infty}) = \{y_i\}_{i=1}^{\infty}$, where

$$y_i = \begin{cases} x_i & \text{if } i \leqslant n, \\ x_{i-n} & \text{if } n < i \leqslant 2n, \\ 0 & \text{if } i > 2n. \end{cases}$$

Since $\lim_{n\to\infty} \|S_n^*(y)\| = 2^{1/q}\|y\|$ for each $y \in l_q$, $\{S_n^*\}_{n=1}^{\infty}$ is not an S.D.O.B. for l_q.

Added in proof. 1. It is shown in [3] that every separable reflexive Banach space which admits an S.O.B. also admits a finite-dimensional Schauder decomposition. 2. The techniques of [3] yield the theorem of this note under the less restrictive hypothesis that X admits a weak S.D.O.B. $\{S_d\colon d \in D\}$. Indeed, one then picks $\{d_1 < d_2 < d_3 < \ldots\} \subset D$ and $\{T_n\}_{n=1}^{\infty}$ with each T_n a convex combination of $\{d\colon d_n \leqslant d \leqslant d_{n+1}\}$ so that $\{T_n(x)\}_{n=1}^{\infty}$ converges strongly to x for each $x \in X$. Then $\{S_{d_{n+1}} + T_n - S_{d_{n+1}} T_n\}_{n=1}^{\infty}$ is an S.D.O.B. for X.

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