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ON ADDITIVE FUNCTIONS WITH A NON-DECREASING NORMAL ORDER

BY

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1. If $f(n)$ is any arithmetical function, then one says (cf. [1], p. 356) that the function $g(n)$ is a *normal order for* $f(n)$ provided, for every positive ε, the inequality

$$|f(n)-g(n)| \leqslant \varepsilon g(n)$$

holds for almost all numbers n, that is, for all n's with the exception of a set of zero density. From the Turán-Kubilius inequality (see, e.g., [2]) one infers that, for a large class of additive functions, one can find normal orders which are non-decreasing. For example, if $f(n) \geqslant 0$ is strongly additive, i.e.,

$$f(n) = \sum_{p|n} f(p)$$

(where the letter p is restricted to prime numbers), and

$$A_N = \sum_{p\leqslant N} f(p)p^{-1}, \quad B_N = \Big(\sum_{p\leqslant N} f^2(p)p^{-1}\Big)^{1/2},$$

then $f(n)$ has a non-decreasing normal order A_n provided $B_N = o(A_N)$ and, for every positive ε, the inequality $|A_n - A_N| < \varepsilon A_N$ holds for $\big(1+o(1)\big)N$ numbers $n \leqslant N$.

These conditions do not form a set of necessary and sufficient conditions, as, e.g., the function

$$f(n) = \sum_{p|n} \log p$$

violates the first of them but, nevertheless, it has a non-decreasing normal order $g(n) = \log n$ (see, e.g., [2], p. 43). On the other hand, we shall show later that, for a positive c, the function

$$f(n) = \sum_{p|n} \log^{1+c} p$$

cannot have a non-decreasing normal order. This makes the following conjecture plausible:

If $H(x)$ is positive and non-decreasing and

$$f(n) = \sum_{p|n} H(p)$$

has a non-decreasing normal order, then one must have $H(x) = O(\log^{1+\varepsilon} x)$ for every positive ε. (**P 923**)

We are unable to settle this conjecture but we shall prove (Theorem I) that under these assumptions one has

$$H(x) = O(\exp\{a \log\log x \log\log\log x\}),$$

where $a > 0$ is arbitrary.

If we, however, make the additional restriction

(i) $H(x)/\log x$ is non-decreasing,

then we can prove this conjecture and even give an intrinsic characterization of those functions $H(x)$ for which $f(n)$ has a non-decreasing normal order (Theorem II). From this characterization it follows that in the statement of the conjecture one cannot remove the ε.

2. We start with two simple lemmas.

LEMMA 1. *Let $f(n)$ be an arbitrary arithmetical function. Assume that there exist two sets A and B of natural numbers, A of positive upper density and B of positive lower density, and two non-decreasing functions $F(x)$ and $G(x)$ such that $f(n) \leqslant F(n)$ for $n \in A$, and $f(n) \geqslant G(n)$ for $n \in B$. If, for every positive M, one can find a certain $\vartheta = \vartheta(M) < 1$ and a set X of natural numbers of positive density such that $F(Mx) < \vartheta G(x)$ for $x \in X$, then $f(n)$ cannot have a non-decreasing normal order.*

The assertion of the lemma remains true if we assume that A and B have a positive lower density and X is infinite.

Proof. Assume that $g(n)$ is a non-decreasing normal order for $f(n)$ and let

$$X_\varepsilon = \{n:\ |f(n) - g(n)| < \varepsilon g(n)\}.$$

Density of this set equals 1, and so $A \cap X_\varepsilon$ is infinite and $B \cap X_\varepsilon$ has a positive lower density. Choose a sufficiently large $a \in A \cap X_\varepsilon$. There are numbers m_1 and m_2 such that in the interval $(a/m_1, a)$ there is an element b of $B \cap X_\varepsilon$, and in the interval $(b/m_2, b)$ an element x of X. Putting $M = m_1 m_2$, we get $x \leqslant b \leqslant a \leqslant Mx$. Now $g(b) \leqslant g(a)$ and thus

$$(1+\varepsilon)^{-1} G(b) \leqslant (1+\varepsilon)^{-1} f(b) \leqslant g(b) \leqslant g(a) \leqslant (1-\varepsilon)^{-1} f(a) \leqslant (1-\varepsilon)^{-1} F(a)$$

which implies

$$G(b) \leqslant (1+\varepsilon)(1-\varepsilon)^{-1} F(a) \leqslant (1+\varepsilon)(1-\varepsilon)^{-1} F(Mx) \leqslant (1+\varepsilon)(1-\varepsilon)^{-1} \vartheta G(x)$$
$$\leqslant (1+\varepsilon)(1-\varepsilon)^{-1} \vartheta G(b) < G(b),$$

provided ε is sufficiently small, and this is a contradiction.

If A and B are of positive lower density and X is infinite, first choose a sufficiently large $x \in X$, then $b \in B \cap X_\varepsilon$ lying in a certain interval $(x, m_1 x)$ and, finally, $a \in A \cap X_\varepsilon$ lying in the interval $(b, m_2 b)$. Put $M = m_1 m_2$ and proceed as above.

LEMMA 2. *Let $q(n)$ be the maximal prime divisor of the number n. Let $A = \{n: q(n) \leqslant n^\varepsilon\}$ and $B = \{n: q(n) \geqslant n^c\}$, where ε is positive, and c is positive and less than* 1. *Then both sets A and B have a positive lower density.*

In fact, better results are known and can be found, e.g., in [3].

3. Now we can prove the following result:

THEOREM I. *Let $H(x)$ be positive and non-decreasing and assume that*

$$f(n) = \sum_{p|n} H(p)$$

has a non-decreasing normal order. Then, for every positive a, we have

$$H(x) = O(\exp\{a \log\log x \log\log\log x\}).$$

Proof. Let A' denote the subset of the set A, occurring in Lemma 2, consisting of all numbers n of A which have at most $(1+\lambda)\log\log n$ distinct prime divisors, where λ is a given positive number. By the Hardy-Ramanujan theorem (see [1], Theorem 431), the set A' has a positive lower density. We apply Lemma 1 to the sets A' and B. We have

$$f(n) \leqslant (1+\lambda)\log\log n\, H(n^\varepsilon) \quad \text{for } n \in A,$$

and

$$f(n) \geqslant H(n^c) \quad \text{for } n \in B.$$

By Lemma 1, we infer that there exists a number M such that, for every $\vartheta < 1$ and all large x, we have

$$\log\log(Mx) H(M^\varepsilon x^\varepsilon) \geqslant \vartheta H(x^c)$$

(here the factor $1+\lambda$ has been covered by ϑ). This inequality leads now directly to our assertion. Indeed, writing $\gamma = c/\varepsilon$, $\beta = M^{-c}$ and $X = (Mx)^\varepsilon$, we obtain

$$H(\beta X^\gamma)/H(X) \leqslant \vartheta^{-1} \log\log X^{1/\varepsilon}. \tag{1}$$

Now let X be sufficiently large so that (1) is satisfied. Put $T_1 = X$ and $T_{k+1} = \beta T_k^\gamma$. Then

$$H(T_{k+1}) = \frac{H(T_{k+1})}{H(T_k)} \cdots \frac{H(T_2)}{H(T_1)} H(T_1) \leqslant H(X)\vartheta^{-k} \prod_{j=1}^{k} \log\log(T_j^{1/\varepsilon}).$$

Since $T_j = X^{\gamma^{j-1}}\beta^{1+\gamma+\ldots+\gamma^{j-2}}$, we obtain easily

$$\log\log(T_j^{1/\varepsilon}) \leqslant j\log\gamma + \log\log(\beta X) - \log\varepsilon,$$

whence

$$H(T_{k+1}) \leqslant H(X)\vartheta^{-k}\prod_{j=1}^{k}\bigl(j\log\gamma + \log\log(\beta X) - \log\varepsilon\bigr).$$

Observe now that, for large j and for an arbitrary $\mu > 0$, we have

$$j\log\gamma + \log\log(\beta X) - \log\varepsilon \leqslant (1+\mu)j\log\gamma$$

and so, finally,

$$\begin{aligned}(2)\qquad H(T_{k+1}) &\leqslant C_1\vartheta^{-k}H(X)(1+\mu)^k(\log^k\gamma)k^k \\ &= C_2\exp\bigl\{k\log k + k\bigl(\log\log\gamma + \log(1+\mu) - \log\vartheta\bigr)\bigr\},\end{aligned}$$

where C_1 and C_2 are appropriate constants. Note now that if U is a given large number and

$$k > 1 + \bigl(\log\log U - \log\log(\beta X)\bigr)/\log\gamma,$$

then $U \leqslant T_{k+1}$, whence $H(U) \leqslant H(T_{k+1})$, and now (2) implies our assertion.

We turn now to functions

$$f(n) = \sum_{p|n} H(p)$$

with $H(x)$ non-negative and such that $h(x) = H(x)/\log x$ is non-decreasing. In this case we can give a rather simple characterization of the functions $H(x)$ for which $f(n)$ has a non-decreasing normal order.

THEOREM II. *Assume that $H(x)$ is non-negative, non-decreasing and satisfies* (i). *Then $f(n)$ has a non-decreasing normal order if and only if $H(x) = \log x\cdot L(\log x)$, where $L(t)$ is slowly oscillating in the sense of Karamata (i.e., $L(2x)/L(x)$ tends to unity if x tends to infinity). This condition is equivalent to each of the following two:*

$$(3)\qquad \sum_{p\leqslant x} H(p)p^{-1} = \bigl(1+o(1)\bigr)H(x),$$

$$(4)\qquad \lim_{x\to\infty} h(x^2)/h(x) = 1.$$

Proof. We show first that (3) and (4) are equivalent (under condition (i)). If we assume (3), then we can write

$$\begin{aligned}\bigl(1+o(1)\bigr)h(x)\log x &= \sum_{p\leqslant x}\bigl(h(p)\log p\bigr)/p \\ &= \sum_{p\leqslant\sqrt{x}}\bigl(h(p)\log p\bigr)/p + \sum_{\sqrt{x}<p\leqslant x}\bigl(h(p)\log p\bigr)/p \\ &\leqslant \tfrac{1}{2}h(\sqrt{x})\log x + O\bigl(h(\sqrt{x})\bigr) + \tfrac{1}{2}h(x)\log x + O\bigl(h(x)\bigr) \\ &= \tfrac{1}{2}\bigl(h(\sqrt{x}) + h(x)\bigr)\log x + O\bigl(h(x)\bigr)\end{aligned}$$

and this is possible only if

$$\lim_{x\to\infty} h(\sqrt{x})/h(x) = 1.$$

Thus (4) holds.

Conversely, if $h(x)$ satisfies (i) and (4), then, for every positive ε,

$$\lim_{x\to\infty} h(x^\varepsilon)/h(x) = 1.$$

Thus

$$h(x)\log x + O(h(x)) \geqslant \sum_{p\leqslant x} (h(p)\log p)/p$$
$$\geqslant \sum_{x^\varepsilon<p\leqslant x} (h(p)\log p)/p \geqslant h(x^\varepsilon)((1-\varepsilon)\log x + O(1)),$$

whence

$$\limsup_{x\to\infty} \Big(\sum_{p\leqslant x} (h(p)\log p)/p\Big)/h(x)\log x \leqslant 1$$

and

$$\liminf_{x\to\infty} \Big(\sum_{p\leqslant x} (h(p)\log p)/p\Big)/h(x)\log x \geqslant 1-\varepsilon.$$

This, obviously, proves (3).

It follows evidently from (4) that the function $L(t) = h(\exp t)$ is slowly oscillating, and so (4) implies $H(x) = \log x\cdot L(\log x)$ with slowly oscillating $L(t)$.

Observe also that the function $H(x)$ is itself non-decreasing and, for every n, we have

$$f(n) = \sum_{p|n} h(p)\log p \leqslant h(n)\sum_{p|n}\log p \leqslant H(n).$$

Now assume that (3) is satisfied. Then

$$\sum_{n\leqslant x} (H(x)-f(n)) = xH(x) - \sum_{n\leqslant x} f(n)$$
$$= xH(x) - \sum_{p\leqslant x} H(p)[x/p] = xH(x) - x\sum_{p\leqslant x} H(p)p^{-1} + O(xH(x)/\log x)$$
$$= o(xH(x)) + O(xH(x)/\log x) = o(xH(x)).$$

If now $N_\eta(x)$ is the number of $n\leqslant x$ with $H(x)-f(n)\geqslant \eta H(x)$ (observe that $H(x)-f(n)$ is always non-negative!), then we get $N_\eta(x) = o(x)$. By virtue of

$$0\leqslant H(n)-f(n)\leqslant H(x)-f(n),$$

we see that $H(n)$ is a non-decreasing normal order for $f(n)$. Finally, assume that $f(n)$ has a non-decreasing normal order. Fix positive ε, $\eta<1/2$ and put

$$A = \{n:\ q(n)\leqslant n^\varepsilon\} \quad\text{and}\quad B = \{n:\ q(n)\geqslant n^{1-\eta}\}.$$

Both sets A and B have, by Lemma 2, a positive lower density. We get

$$f(n) = \sum_{p|n} h(p)\log p \leqslant h(n^{\varepsilon})\log n \quad \text{for } n \in A,$$

and

$$f(n) \geqslant (1-\eta)\log n\, h(n^{1-\eta}) \quad \text{for } n \in B.$$

Thus Lemma 1 implies that, for a certain M, the inequality

$$h(Mx^{\varepsilon})/h(x^{1-\eta}) > \vartheta(1-\eta)$$

holds for every $\vartheta < 1$ and sufficiently large x. Thus we obtain

$$h(x^{2\varepsilon})/h(x^{1/2}) \geqslant h(x^{2\varepsilon})/h(x^{1-\eta}) > \vartheta(1-\eta) \quad \text{for } x \geqslant x_0(\eta, \vartheta, \varepsilon),$$

whence

$$\liminf_{x\to\infty} h(x^{2\varepsilon})/h(x^{1/2}) \geqslant \vartheta(1-\eta).$$

But this must hold for any $\vartheta < 1$ and $\eta > 0$, and so we, finally, arrive at

$$\lim_{x\to\infty} h(x^{2\varepsilon})/h(x^{1/2}) = 1$$

which is equivalent to (3).

COROLLARY. *If $H(x)$ is non-negative, $H(x)/\log x$ is non-decreasing and*

$$f(n) = \sum_{p|n} H(p)$$

has a non-decreasing normal order, then $H(x) = O\big((\log x)^{1+\varepsilon}\big)$ for every positive ε.

Proof. It suffices to observe that every function $h(x)$ satisfying (3) satisfies also $h(x) = O\big((\log x)^{\varepsilon}\big)$ for every positive ε.

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