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MULTIPLIERS ON SCHWARTZ SPACES OF SOME LIE GROUPS

BY

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1. Introduction. Let $S(G)$ denote the suitably defined Schwartz space of a Lie group G. If G is nilpotent, then $S(G)$ is defined as the usual Schwartz space of $\boldsymbol{R}^n$ [1]. Let $S'(G)$ denote the space of tempered distributions on G. We say that $T \in S'(G)$ is a *left* S-*multiplier* on G if $T * \varphi \in S(G)$ whenever $\varphi \in S(G)$, and that T is a *right* S-*multiplier* if $\varphi * T \in S(G)$ whenever $\varphi \in S(G)$. T is called an S-*multiplier* if it is both a left and a right S-multiplier.

It is known that $T \in S'(\boldsymbol{R}^n)$ is an S-multiplier if, and only if, $\hat{T}$ and all its derivatives are slowly increasing C^∞ functions.

S-multipliers play an important role in the study of harmonic analysis on nilpotent Lie groups [1].

S-multipliers on the Heisenberg groups were characterized in [1]. Our main goal is to characterize left S-multipliers on the motion groups and to provide explicit examples of right S-multipliers which fail to be left S-multipliers on some classical Lie groups. Eymard asked for such examples of L^p-multipliers on non-commutative locally compact groups [2]. A. M. Mantero recently answered Eymard's question for the Heisenberg group [3]. There is a strong structural similarity between our examples of Schwartz multipliers and those of A. M. Mantero.

2. Preliminaries and notation. Let $\boldsymbol{T}$ denote the circle group. $\boldsymbol{T}$ may be identified with the complex numbers of modulus 1. The motion group of the plane is the semi-direct product $M(2) = \boldsymbol{T} \times \boldsymbol{C}$ where $\boldsymbol{C}$ denotes the additive group of complex numbers. The multiplication law in $M(2)$ is given by

$$(e^{i\alpha}, z)(e^{i\beta}, w) = (e^{i(\alpha+\beta)}, z + e^{i\alpha} w).$$

Let H_n denote the $2n+1$ dimensional Heisenberg group which consists of elements (x, y, z), $x, y \in \boldsymbol{R}^n$, $z \in \boldsymbol{R}$, with multiplication defined by

$$(x, y, z)(\alpha, \beta, \gamma) = (x+\alpha, y+\beta, z+\gamma+\langle \beta, x\rangle)$$

(Here, $\langle\ ,\ \rangle$ is the usual inner product on $\boldsymbol{R}^n$.) Let H_n^k denote the Heisenberg group with compact center. The elements of H_n^k are (x, y, e^{iz}), $x, y \in \boldsymbol{R}^n$, $e^{iz} \in \boldsymbol{T}$ with multiplication defined by

$$(x, y, e^{iz})(\alpha, \beta, e^{i\gamma}) = (x+\alpha, y+\beta, e^{i(z+\gamma+\langle x,\beta\rangle)}).$$

We denote by $S(M(2))$ the space of functions satisfying

$$\sup_z (1+|z|^k) \left| \frac{\partial^n}{\partial \alpha^n} D^{m_1,m_2} \psi \right| < \infty$$

for all $m_1, m_2, k \geqslant 0$ where

$$D^{m_1,m_2}\psi = \frac{\partial^{m_1+m_2}\psi}{\partial x^{m_1}\, \partial y^{m_2}}.$$

By $S'(G)$ we denote the dual space of $S(G)$ which consists of the tempered distributions on G.

For $T \in S'(G)$ where G is an Abelian Lie group, we denote by $\hat{T}$ the Fourier transform of T and by $\hat{G}$ the character group of G.

One notices that for a unimodular G we have $(T * \varphi)(g) = \tilde{\psi}(g)$ where $\psi(g) = \int_G \varphi(gg')\, T(g')\, d\mu(g')$ and $\tilde{\psi}(g) = \psi(g^{-1})$. For the motion group and the Heisenberg groups we have $\psi \in S$ iff $\tilde{\psi} \in S$. Thus T is a left S-multiplier on these groups if $\psi \in S(G)$ whenever $\varphi \in S(G)$. Similarly, T is a right S-multiplier if $\theta \in S(G)$ whenever $\varphi \in S(G)$ where

$$\theta(g) = \int_G T(g'g)\, \alpha(g')\, d\mu(g').$$

3. S-multipliers for $M(2)$. The characterization of left S-multipliers on the motion group is provided by

THEOREM 1. *The tempered distribution $T \in S'(M(2))$ is a left S-multiplier on $M(2)$ if, and only if, T is an S-multiplier for $\boldsymbol{T} \times \boldsymbol{C}$. Namely, $\hat{T}$ and all its derivatives are C^∞ slowly increasing functions on $\boldsymbol{Z} \times \hat{\boldsymbol{C}}$.*

Proof. Suppose that T is an S-multiplier for $\boldsymbol{T} \times \boldsymbol{C}$. Let $\varphi \in S(M(2))$ and let ψ be defined by

$$\psi(g) = \int_{M(2)} \varphi(gg')\, T(g')\, dg', \quad g \in M(2).$$

Hence

$$\psi(e^{i\alpha}, z) = \int_{M(2)} \varphi\big(e^{i(\alpha+\theta)}, e^{i\alpha}(e^{-i\alpha} z + w)\big)\, T(e^{i\theta}, w)\, d\mu \tag{1.1}$$

where $g = (e^{i\alpha}, z)$. For $e^{i\alpha} \in \boldsymbol{T}$ let $\varphi_\alpha \in S(M(2))$ be defined by

$$\varphi_\alpha(e^{i\xi}, z) = \varphi(e^{-i\xi}, -e^{i\alpha} z). \tag{1.2}$$

Thus we have

$$\psi(e^{i\alpha}, z) = (\varphi_\alpha * T)(e^{-i\alpha}, -e^{-i\alpha} z)$$

where $\varphi_\alpha * T$ is the convolution in $T \times C$. Consequently, $\psi(e^{i\alpha}, \cdot) \in S(C)$ for each $e^{i\alpha} \in T$, implying, by the compactness of T, that $\sup_{T \times C} |(1+|z|^2)^N D^{m_1, m_2} \psi| < \infty$ for all positive integers m_1, m_2 and N.

It remains to show that $\frac{\partial^n}{\partial \alpha^n} D^{m_1, m_2} \psi$ is rapidly decreasing for each $n \geqslant 0$. For $n = 1$, we have

$$\frac{\partial}{\partial \alpha} D^{m_1, m_2} \psi(e^{i\alpha}, z)$$

$$= \int_{M(2)} \frac{\partial}{\partial \alpha} D^{m_1, m_2} \varphi(e^{i(\alpha+\theta)}, z + e^{i\alpha} w)\, T(e^{i\theta}, w)\, d\mu$$

$$+ \int_{M(2)} D^{m_1+1, m_2} \varphi(e^{i(\alpha+\theta)}, z + e^{i\alpha} w)\, R_e(iwe^{i\alpha})\, T(e^{i\theta}, w)\, d\mu$$

$$+ \int_{M(2)} D^{m_1, m_2+1} \varphi(e^{i(\alpha+\theta)}, z + e^{i\alpha} w)\, I_m(iwe^{i\alpha})\, T(e^{i\theta}, w)\, d\mu.$$

It follows, therefore, that $\frac{\partial^n}{\partial \alpha^n} D^{m_1, m_2} \psi$ is a sum of expressions of the form

$$\chi_k(e^{i\alpha}, z) = \int_{M(2)} \varphi_k(e^{i(\alpha+\theta)}, z + e^{i\alpha} w)\, P_k(e^{i\alpha}, w)\, T(e^{i\theta}, w)\, d\mu$$

where $\varphi_k \in S(m(2))$ and $P_k(e^{i\alpha}, \cdot)$ is a polynomial in $R_e w$ and $I_m w$ for each fixed $e^{i\alpha}$. Hence, arguing as in the case $n = 0$, we conclude that each term is rapidly decreasing which verifies the "if" part of the theorem.

Suppose now that $T \in S'(M(2))$ is a left S-multiplier on $M(2)$. Let $S^{(r)}(M(2)) \subset S(M(2))$ consist of φ such that $\varphi(e^{i\alpha}, \cdot)$ is a radial function on C for each $e^{i\alpha} \in T$. If $\varphi \in S^{(r)}(M(2))$, then $\varphi_a(e^{i\xi}, z) = \varphi(e^{-i\xi}, -z)$ for each $e^{i\alpha} \in T$. Therefore, by (1.1) and (1.2), $\psi \in S(M(2))$ where $\psi(e^{i\alpha}, z) = (\varphi * T)(e^{-i\alpha}, -e^{-i\alpha} z)$ and $\varphi * T$ is the commutative convolution in $T \times C$. Because $\tilde{\psi} \in S(M(2))$, where $\tilde{\psi}(g) = \psi(g^{-1})$, we must have $\varphi * T \in S(M(2))$ (the convolution in $T \times C$). Hence $\hat{T}$ is a C^∞ function on $Z \times \hat{C}$, and $\hat{T}\hat{\varphi} \in S(Z \times \hat{C})$ for $\varphi \in S^{(r)}(M(2))$. One notices also that $(D^{m_1, m_2} \hat{T})\hat{\varphi} \in S(Z \times \hat{C})$, for $\varphi \in S^{(r)}(M(2))$ and $m_1, m_2 \geqslant 0$, which implies that $D^{m_1, m_2} \hat{T}$ is slowly increasing. That completes the proof of the theorem.

S-multipliers for C may be lifted to $M(2)$ as described in

THEOREM 2. *Let $T \in S'(C)$ and $P = 1 \otimes T \in S'(M(2))$. Then P is an S-multiplier on $M(2)$ if, and only if, T is an S-multiplier on C.*

Proof. Suppose that $T \in S'(C)$ is an S-multiplier on C. Let $\tilde{p} \in S'(M(2))$ be defined as $\tilde{p}(g) = p(g^{-1})$ where $p = 1 \otimes T$. We have $\tilde{p}(e^{i\alpha}, z)$

$= T(-e^{-i\alpha}z)$ and by a direct computation we obtain

$$\hat{\tilde{p}}(m, \lambda) = \int_{T} \hat{T}(-e^{-i\alpha}\lambda) e^{-im\alpha} d\alpha, \quad (m, \lambda) \in Z \times \hat{C},$$

or

$$\hat{\tilde{p}}(m, \lambda) = (-1)^m e^{-im\theta} \int_0^{2\pi} \hat{T}(r, \xi) e^{im\xi} d\xi$$

where $\lambda = (r, \theta)$ in polar coordinates. Since T is an S-multiplier on C, $\hat{T}$ and all its derivatives are slowly increasing C^∞ functions on C. This implies that $\hat{\tilde{P}}$ and all its derivatives (with respect to r and θ) are slowly increasing functions on $Z \times C$. It follows by Theorem 1 that $\tilde{P}$ is a left S-multiplier on $M(2)$ and that P is a right S-multiplier. However, by Theorem 1, P is a left S-multiplier on $M(2)$ if, and only if, T is an S-multiplier on C. That completes the proof.

Remark 3. The simple characterization of left S-multipliers on $M(2)$ is mainly due to the compactness of T. For the Heisenberg groups the characterization is more intricate [1]. The analogous of Theorem 1 is false even for the three-dimensional Heisenberg group with compact center. (See Remark 10.)

Remark 4. The analogue of Theorem 1 for L_2-multipliers on $M(2)$ is false. Actually, we have the following: Let $T \in S'(C)$ and $P = 1 \otimes T$. Then P is an L_2-multiplier on $M(2)$ if, and only if, $\sup_r \int_0^{2\pi} |\hat{T}(r, \theta)|^2 d\theta < \infty$. Consequently, there exist L_2-multipliers on $M(2)$ which are not pseudo-measures on $T \times C$.

4. Right and left S-multipliers. In this section, we introduce right S-multipliers on some Lie groups which fail to be left S-multipliers. We start with the motion group. If $1 \otimes T$, $T \in S'(C)$, is a left S-multiplier on $M(2)$ then, by Theorem 2, $1 \otimes T$ is also a right S-multiplier. However, the converse is false as described in

THEOREM 5. *Let $P \in S'(C)$ such that $\hat{P}(r, \theta) = g(r) h(\theta)$ where $g \in C^\infty(R)$, $\operatorname{supp} g \subset [a, b]$, $a > 0$, and h is a 2π-periodic measure. Then $T = 1 \otimes P$ is a right S-multiplier on $M(2)$. If h is not a C^∞ function then T is not a left S-multiplier.*

Proof. Let $\varphi \in S(M(2))$ and let $\psi(g) = \int_{M(2)} T(g'g) \varphi(g'g) dg'$. We have

$$\psi(e^{i\alpha}, z) = \int_{M(2)} P(e^{i\theta} z + w) \varphi(e^{i\theta}, w) d\mu.$$

For $e^{i\theta}\in \boldsymbol{T}$ let φ_θ be defined by $\varphi_\theta(z)=\varphi(e^{i\theta},\,-z)$. Hence

$$\psi(e^{i\alpha},z)=\psi_1(z)=(2\pi)^{-1}\int_0^{2\pi}(P*\varphi_\theta)(-e^{i\theta}z)\,d\theta$$

where $P*\varphi_\theta$ is the convolution on $\boldsymbol{C}$. Taking a Fourier transform, we obtain

$$\begin{aligned}\hat{\psi}_1(\xi)&=(2\pi)^{-1}\int_0^{2\pi}\hat{P}(e^{-i\theta}\xi)\,\hat{\varphi}_\theta(e^{-i\theta}\xi)\,d\theta\\&=g(r)(2\pi)^{-1}\int_0^{2\pi}h(\alpha-\theta)\,\hat{\varphi}_\theta(r,\,\alpha-\theta)\,d\theta\\&=g(r)(2\pi)^{-1}\int_0^{2\pi}\hat{\varphi}_{\alpha-\theta'}(r,\,\theta')\,h(\theta')\,d\theta'\end{aligned}$$

where $\xi=(r,\,\alpha)$ denotes polar coordinates in $\boldsymbol{C}$.

Notice that for each $\theta'\in[0,\,2\pi)$ the function $\chi_{\theta'}(r,\,\alpha)=\hat{\varphi}_{\alpha-\theta'}(r,\,\theta')$ is smooth on $\boldsymbol{C}$. It follows, therefore, that $\hat{\psi}_1\in S(\boldsymbol{C})$ implying that $\psi_1\in S(\boldsymbol{C})$, as required. By Theorem 2, T is a left S-multiplier if, and only if, h is a C^∞ 2π-periodic function, and the proof is completed.

In particular, for $h=\delta_0$ we obtain

COROLLARY 6. *Let $g\in C^\infty(\boldsymbol{R})$ such that $\hat{g}$ is a C^∞ function supported on $[a,\,b]$, $a>0$. Let $P\in C^\infty(\boldsymbol{C})$ be defined by $P(z)=g(\mathrm{Re}(z))$. Then $T=1\otimes P$ is a right S-multiplier which is not a left S-multiplier on $M(2)$.*

Let $H_1=\boldsymbol{R}\times\boldsymbol{R}^2$ be the three-dimensional Heisenberg group. We recall that the non-trivial orbits in $\hat{\boldsymbol{R}}^2$ under the action of $\boldsymbol{R}$ are the lines $\{(y,\,z)\in\boldsymbol{R}^2\colon\, y\in\boldsymbol{R}\}$. Hence the functions of z only are the suitable "radial" functions for H_1. On the other hand a distribution of the form $1\otimes P$, $P\in S'(\boldsymbol{R}^2)$, is neither a right nor a left S-multiplier H_1.

However, we may state the following analogue of Theorem 5.

THEOREM 7. *Let $h,\,g\in C^\infty(\boldsymbol{R})$, $R\in S(\boldsymbol{R})$ such that* $\operatorname{supp}\hat{g}\subseteq[a,\,b]$, $a>0$ *and $\hat{h}$ a compactly supported measure on $\boldsymbol{R}$. Let $T\in S'(H_1)$ where $T(x,\,y,\,z)=R(x)h(y)g(z)$. Then T is a right S-multiplier on H_1. If $\hat{h}$ is not a C^∞ function then T is not a left S-multiplier. In particular, the C^∞ function $T(x,\,y,\,z)=R(x)g(z)$ is a right S-multiplier which is not a left S-multiplier.*

Proof. Let $\varphi\in S(H_1)$ and let $\psi(g)=\int T(g'g)\,\varphi(g')\,dg'$. Let $P\in C^\infty(\boldsymbol{R}^2)$ be defined by $P(y,\,z)=h\otimes g(y,\,z)$. Hence, we have

$$\psi(x,\,y,\,z)=\int_{\boldsymbol{R}}R(x+\alpha)\Big[\iint_{\boldsymbol{R}^2}P(y+\beta,\,z+\alpha y+\lambda)\,\varphi(\alpha,\,\beta,\,\lambda)\,d\beta\,d\lambda\Big]\,d\alpha.$$

Let $\varphi_\alpha(\beta,\,\lambda)=\varphi(\alpha,\,-\beta,\,-\lambda)$, $\varphi\in\boldsymbol{R}$. Then

$$\psi(x,\,y,\,z)=\int_{\boldsymbol{R}}R(x+\alpha)\,[(P*\varphi_\alpha)(-y,\,-z-\alpha y)]\,d\alpha$$

where $P * \varphi_\alpha$ is the convolution in $\boldsymbol{R}^2$. One notices that if we have $\chi(y, z) = \varphi(y, z+\alpha y)$ for some fixed $\alpha \in \boldsymbol{R}$, then $\hat{\chi}(\lambda_2, \lambda_3) = \hat{\varphi}(\lambda_2 - \alpha\lambda_3)$, $(\lambda_2, \lambda_3) \in \boldsymbol{R}^2$. Thus we have

$$\hat{\psi}(\lambda_1, \lambda_2, \lambda_3) = \int_{\boldsymbol{R}} \Big[\int_{\boldsymbol{R}} R(x+\alpha) e^{-ix\lambda_1} dx \Big] \hat{P}(\alpha\lambda_3 - \lambda_2, -\lambda_3) \hat{\varphi}_2(\alpha, \alpha\lambda_3 - \lambda_2, -\lambda_3) d\alpha$$

$$= \int_{\boldsymbol{R}} \hat{P}(\alpha\lambda_3 - \lambda_2, -\lambda_3) \hat{\varphi}_{2,3}(\alpha, \alpha\lambda_3 - \lambda_2, -\lambda_3) \hat{R}(\lambda_1) e^{i\alpha\lambda_1} d\alpha$$

$$= e^{i\lambda_2\lambda_1/\lambda_3} \hat{R}(\lambda_1) \frac{\hat{g}(-\lambda_3)}{\lambda_3} \int_{\boldsymbol{R}} \hat{\varphi}_{2,3}\left(\frac{\theta+\lambda_2}{\lambda_3}, \theta, -\lambda_3\right) e^{i\theta\lambda_1/\lambda_3}$$

where

$$\hat{\varphi}_{2,3}(\alpha, \xi_1, \xi_2) = \int_{\boldsymbol{R}^2} \varphi(\alpha, \beta, \lambda) e^{-i(\beta\xi_1 + \lambda\xi_2)} d\beta\, d\lambda.$$

Here $\exp(x) = e^x$. For each $\theta \in \boldsymbol{R}$ the function

$$U_\theta(\lambda_1, \lambda_2, \lambda_3) = \hat{\varphi}_{2,3}\left(\frac{\theta+\lambda_2}{\lambda_3}, \theta, -\lambda_3\right) \exp(i\theta\lambda_1/\lambda_3)$$

is C^∞ and rapidly decreasing in the strip $\{(\lambda_1, \lambda_2, \lambda_3) \in \hat{\boldsymbol{R}}^3 : -b < \lambda_3 < -a\}$. It follows that $\hat{\psi} \in S(\hat{\boldsymbol{R}}^3)$ and hence that $\psi \in S(H_1)$ proving that T is a right S-multiplier on H_1.

Suppose now that T is a left S-multiplier. Let $\varphi \in S(H_1)$ where $\varphi(\alpha, \beta, \lambda) = \varphi_1 \otimes \varphi_2(\alpha, \beta, \lambda)$, $\varphi_1 \in S(\boldsymbol{R})$, $\varphi_2 \in S(\boldsymbol{R}^2)$. Hence $\chi \in S(H_1)$ where

$$\chi(x, y, z) = \int_{\boldsymbol{R}^2} P(y+\beta, z+\beta x+\lambda) \varphi_2(\beta, \lambda) d\beta\, d\lambda \cdot \int_{\boldsymbol{R}} R(x+\alpha) \varphi_1(\alpha) d\alpha.$$

Since $\chi(0, \cdot, \cdot) \in S(\boldsymbol{R}^2)$, it follows that P must be an S-multiplier on $\boldsymbol{R}^2$. Consequently, $\hat{P}$ should be a C^∞ function, a contradiction. Finally, the last statement of the theorem is provided by $h = \delta_0$ which completes the proof.

Similarly we may prove the following for H_1^k.

THEOREM 8. *Let $h \in C^\infty(\boldsymbol{R})$, $R \in S(\boldsymbol{R})$ such that $\hat{h}$ is a compactly supported measure on $\boldsymbol{R}$. Let g be a trigonometric polynomial such that $\hat{g}(0) = 0$. Let $T \in S'(H_1^k)$ be defined by $T(x, y, e^{iz}) = R(x) h(y) g(e^{iz})$. Then T is a right S-multiplier on H_1^k. If $\hat{h}$ is not a C^∞-function then T is not a left S-multiplier. In particular, the function $T(x, y, e^{iz}) = R(x) e^{iz}$ is a right S-multiplier which is not a left S-multiplier on H_1^k.*

Remark 9. Theorems 5 and 7 have an obvious extension to all Euclidean motion groups and to Heisenberg groups. By choosing K-times differentiable functions h we provide $O(1/\|x\|^k)$ functions that are right S-multipliers and fail to be left S-multipliers.

Remark 10. For $D \in S'(\mathbf{R}^2)$ and $\varphi \in S(\mathbf{R}^2)$ let $D ** \varphi$ denote the "twisted convolution" induced by H_1^k on $\mathbf{R}^2$ and defined by $D ** \varphi(x, y) = \int_{\mathbf{R}^2} D(x+\alpha, y+\beta) e^{i\alpha y} \varphi(\alpha, \beta) d\alpha d\beta$. Then $D(x, y) = R(x)$ where $R \in S(\mathbf{R})$ is an S-multiplier with respect to the "twisted convolution" which fails to be an S-multiplier on $\mathbf{R}^2$.

It follows therefore that $D(x, y, e^{iz}) = R(x) e^{iz}$ is a right S-multiplier on H_1^k which is not an S-multiplier on $\mathbf{R}^2 \times \mathbf{T}$, in contrast to Theorem 1 for the motion group.

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