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AN EXTENSION OF THE GENERALIZED BERNSTEIN LEMMA*

BY

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This note is dedicated to Professor Franciszek Leja on the occasion of his 80th anniversary, in recognition of his admirable contributions to the theory of extremal point sets and approximation by polynomials.

The following lemma has shown itself useful [1, 2] in the study of sequences of rational functions in case the finite poles are uniformly bounded. The proposition without that qualification is also true, and its proof is the object of the present note.

We say that a rational function of the form

$$r_{jk}(z) \equiv \frac{a_0 z^j + a_1 z^{j-1} \ldots + a_j}{b_0 z^k + b_1 z^{k-1} \ldots + b_k}, \qquad \sum |b_i| \neq 0,$$

is *of type* (j, k).

LEMMA. *Let E be a closed bounded point set whose complement is connected, and regular in the sense that it possesses a Green's function $G(z)$ with pole at infinity. We denote generically by Γ_σ the locus $G(z) = \log\sigma\,(>0)$ and its interior by E_σ. Let rational functions $r_{n\nu}(z)$ of respective types (n, ν), where ν is constant, satisfy the inequality*

$$(1) \qquad \limsup_{n\to\infty}\,[\max|r_{n\nu}(z)|,\ z \text{ on } E]^{1/n} \leqslant 1/\varrho_1, \qquad 1 < \varrho_1 \leqslant \infty.$$

Let S be a closed set in the closed interior of E_σ, $1 < \sigma < \varrho_1$, and containing no limit point of the poles of the $r_{n\nu}(z)$. Then the sequence $r_{n\nu}(z)$ converges uniformly to zero on S, and we have

$$(2) \qquad \limsup_{n\to\infty}\,[\max|r_{n\nu}(z)|,\ z \text{ on } S]^{1/n} \leqslant \sigma/\varrho_1.$$

The $r_{n\nu}(z)$ need not be defined for every n.

This lemma has already been proved [1, 2] for the case that the finite poles of the $r_{n\nu}(z)$ are uniformly bounded, so we here consider the

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contrary case. Choose a sequence of the $r_{n\nu}(z)$ such that

$$\lim_{n\to\infty}[\max|r_{n\nu}(z)|, z \text{ on } S]^{1/n}$$

equals the first member of (2). Choose a subsequence of that sequence such that at least one finite pole of the successive terms becomes infinite, then (if possible) a subsequence of that subsequence such that at least two finite poles of the successive terms become infinite, and so on as long as possible. Eventually we reach a subsequence such that precisely μ finite poles of the terms become infinite, $1 \leqslant \mu \leqslant \nu$, and the remaining $\nu-\mu$ (or fewer) finite poles are uniformly bounded. For the elements $\psi_n(z)$ of that last sequence we denote by β_j (depending on n) the μ poles of the successive $\psi_n(z)$ that become infinite. Let the point set E_σ and the uniformly bounded poles of the sequence lie in some disk $|z| \leqslant A$, and for each sufficiently large n define

$$\varphi_n(z) \equiv \psi_n(z)\prod_{j=1}^{\mu}\frac{z-\beta_j}{-\beta_j}$$

when the corresponding β_j are in modulus greater than A. The rational function $\varphi_n(z)$ is of type $(n, \nu-\mu)$. It is clear that replacement in the sequence $\psi_n(z)$ [$\equiv r_{n\nu}(z)$]) of $\psi_n(z)$ by $\varphi_n(z)$ does not alter the first member of (1) nor of (2). The replacement sequence $\varphi_n(z)$ has all its finite poles uniformly bounded, so the original proof of the original lemma implies (2) for the subsequence, and hence for the $r_{n\nu}(z)$.

The phraseology of this proof assumes that the $\psi_n(z)$ have effectively ν finite poles, but the only necessary modification for the contrary case is the choice of a suitable ν', $0 \leqslant \nu' < \nu$, and of a suitable subsequence of the $\psi_n(z)$ each of type (n, ν') with effectively ν' finite poles; the reasoning as given is then applicable.

With $\nu = 0$, the Lemma is essentially the generalized Bernstein lemma for polynomials ([3], § 4.6) which has many applications to polynomial approximation. The new lemma has many applications to the study of approximation by rational functions, as the writer plans to indicate elsewhere.

REFERENCES

[1] J. L. Walsh, *The convergence of rational functions of best approximation,* Mathematische Annalen 155 (1964), p. 252-264.

[2] — *The convergence of sequences of rational functions of best approximation with some free poles,* Proceedings of General Motors Symposium of Approximation, September 1964, edited by H. L. Garabedian, published by Elsevier (Amsterdam), 1965, p. 1-16.

[3] — *Interpolation and approximation,* American Mathematical Society Colloquium Publications 20 (1935).

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