COLLOQUIUM MATHEMATICUM

XVI DÉDIÉ À M. FRANCISZEK LEJA 1967

SOME DISTORTION THEOREMS RELATED TO AN INVARIANT METRIC IN C^2

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1. Introduction. The application of the kernel function enables us to define in a (4-dimensional) domain $\mathfrak{d}$ the metric whose line element has the length $ds_{\mathfrak{d}}(z)$,

$$ds_{\mathfrak{d}}^2(z) = \sum_{m,n=1}^{2} T_{m\bar{n}} dz_m d\bar{z}_n, \qquad T_{m\bar{n}} = \frac{\partial^2 \log K_{\mathfrak{d}}}{\partial z_m \partial \bar{z}_n}, \tag{1}$$

and which is invariant (1) with respect to PCT's (pseudo-conformal transformations). Further, this approach permits us to determine bounds for the distortion of Euclidean measures in PCT's. Here $K_{\mathfrak{d}}(z_1, z_2; \bar{z}_1, \bar{z}_2)$ is the kernel function of the domain $\mathfrak{d}$ (see [1]-[8]).

As it has been shown in [9] and [14], the generalization of these methods enables us to give bounds for the distortion of invariant measures in the case of a special class of QPCT's (quasi-pseudo-conformal transformations), mapping a hypersphere onto a class of Reinhardt circular domains.

There arises the question to determine bounds for the distortion of the length (1) in the case of general one-to-one, differentiable transformations with non-vanishing Jacobian, to be called QPCT's W.

Remark 1. While a PCT transforms an infinitesimal hypersphere $\mathfrak{H}$ into a hyperellipsoid of a special type, a differentiable QPCT transforms $\mathfrak{H}$ into a general hyperellipsoid (see Remark p. 938 of [9]).

Since the names PCT and QPCT have been used already previously and since our considerations are closely connected with the theory of analytic functions of several complex variables, we retain these names.

(*) This work was supported in part by National Science Foundation grant GP 5797 at Stanford University.

(1) Here and in the following, invariant = invariant with respect to PCT's.

(2) As a rule, small characters are used for symbols of sets belonging to the original domain $\mathfrak{d}$, and capital characters are used for symbols of sets belonging ot the transformed domain $\mathfrak{D} = W(\mathfrak{d})$.

Remark 2. When considering the domains ([2]) $\mathfrak{d}$ and $\mathfrak{D} = W(\mathfrak{d})$, we often make analogous hypotheses or carry out similar considerations in the case of both domains $\mathfrak{d}$ and $\mathfrak{D}$. To avoid repetition, we formulate the assumptions and our considerations for the domain $\mathfrak{d}$. In paranthe-ses (...) we include the symbols referring to the transformed domain $\mathfrak{D} = W(\mathfrak{d})$. This means that the statement is valid if we replace small characters by capital ones. For instance: $\mathfrak{a}(\mathfrak{A})$ is a circle of radius $\varrho(P)$ means two statements: 1) $\mathfrak{a}$ is a circle of radius ϱ, and 2) $\mathfrak{A}$ is a circle of radius P.

In the following we shall assume that the QPCT W defined in a (bounded and closed) domain $\bar{\mathfrak{d}}$ possesses the following property: the Euclidean length $L(p_1, p_2)$ between two arbitrary points p_k, $p_k \in \bar{\mathfrak{d}}$, and the Euclidean length $L(W(p_1), W(p_2))$ between the transformed points, satisfy the relation

$$0 < \frac{1}{e} \leqslant \frac{L(W(p_1), W(p_2))}{L(p_1, p_2)} \leqslant e. \tag{2}$$

Remark 3. It would be more proper to call W an e-QPCT, but for shortness we omit e.

Let $du = (du_1, du_2)$ at the point (z_1, z_2) be an (infinitesimal) ([3]) vector tangent to the curve $\mathfrak{g}^1\{p_1(s), p_2(s)\}$, $0 \leqslant s \leqslant s_0$, where $p_\varkappa(s)$, $\varkappa = 1, 2$, are differentiable. Further, let $W = \{w_1(z_1, z_2), w_2(z_1, z_2)\}$, where $w_\varkappa$ are differentiable in $\mathfrak{d}$. Then

$$dU = \left\{\left[\sum_{\varkappa=1}^{2}\left(\frac{\partial w_1}{\partial p_\varkappa}p'_\varkappa(s) + \frac{\partial w_1}{\partial \bar{p}_\varkappa}\bar{p}'_\varkappa(s)\right)\right]ds,\ \left[\sum_{\varkappa=1}^{2}\left(\frac{\partial w_2}{\partial p_\varkappa}p'_\varkappa(s) + \frac{\partial w_2}{\partial \bar{p}_\varkappa}\bar{p}'_\varkappa(s)\right)\right]ds\right\}$$

is the image $W(du)$ of du. The problem is to find sufficient conditions for $\mathfrak{d}$, $\mathfrak{D} = W(\mathfrak{d})$, and du so that (2) implies the inequality

$$0 < \frac{1}{c} \leqslant \frac{ds^2_{\mathfrak{d}}(z, \bar{z}, du, d\bar{u})}{ds^2_{\mathfrak{D}}(Z, \bar{Z}, dU, d\bar{U})} \leqslant c. \tag{3}$$

Here $z = (z_1, z_2)$ is a point of $\mathfrak{d}$, Z its image in $\mathfrak{D}$; dU is the image of the vector du. In the case of one variable one easily obtains sufficient conditions of this type. In this case ([4])

$$ds^2_{\mathfrak{d}}(z, \bar{z}, du, d\bar{u}) = K_{\mathfrak{d}}(z, \bar{z})|du|^2, \tag{4}$$

$$ds^2_{\mathfrak{D}}(Z, \bar{Z}, dU, d\bar{U}) = K_{\mathfrak{D}}(Z, \bar{Z})|dU|^2. \tag{5}$$

([3]) In the future the word "infinitesimal" will be omitted.

([4]) If $W = w(z)$, it is sufficient that $\operatorname{Re} w(z)$ and $\operatorname{Im} w(z)$ are differentiable functions of x and y.

Since we assume that

$$(6) \qquad 0 < \frac{1}{e} \leqslant \frac{|du|}{|dU|} \leqslant e,$$

it is sufficient to determine conditions which imply the existence of a constant c such that

$$(7) \qquad 0 < \frac{1}{c} \leqslant \frac{K_{\mathfrak{d}}(z, \bar{z})}{K_{\mathfrak{D}}(Z, \bar{Z})} \leqslant c, \qquad Z = W(z),$$

holds for $z \in \mathfrak{d}$. Since the kernel function $K_{\mathfrak{d}}(z, \bar{z})$ is an analytic non-vanishing function of $z, \bar{z}$, for $z \in \mathfrak{d}$, inequality (7) holds for every transformation W and for points z and $Z = W(z)$ which have some fixed distance, say ϱ, $\varrho > 0$, from the boundary. Further, from (2) follows: if the point z has the distance ϱ from the boundary, the corresponding point $Z = W(z)$ has at least the distance ϱ/e from the boundary of $\mathfrak{D}$. Hence (7) holds for all points z of $\mathfrak{d}$ which have a distance $\varrho \geqslant \varrho_0$, $\varrho_0 > 0$, from the boundary. Consequently, it is sufficient to derive bounds for any sequence of points $z_\varkappa$ such that $L(z_\varkappa, o) \to 0$, for $\varkappa = 1, 2, \ldots$, where o is a boundary point. We shall formulate sufficient conditions for domains $\mathfrak{d}$ and $\mathfrak{D}$, insuring inequality (7) for a sequence of points [5] $z_\varkappa \to o$ and $Z_\varkappa = W(z_\varkappa) \to O = W(o)$, where $o \in \partial\mathfrak{d}$, $O \in \partial\mathfrak{D}$, $\mathfrak{D} = W(\mathfrak{d})$.

At first we assume that the sequence of the points converges to the boundary point o, while $Z_\varkappa = W(z_\varkappa)$ converges to the point O, and we assume that two circles of radius $\varrho(P)$, $\varrho > 0\,(P > 0)$, exist, the first with the center on the interior normal, the second with the center on the exterior normal, whose boundaries have only the point o (O) in common with $\partial\mathfrak{d}$ $(\partial\mathfrak{D})$. Further we assume that the set $\{z_\varkappa\}$ $(\{Z_\varkappa\})$ lies in the angular domain

$$(8) \qquad \Omega_c = \left[0 < c \leqslant \frac{\operatorname{Re}(z_\varkappa - z^0)}{|z_\varkappa - z^0|}\right] \quad \left(\Omega_c = \left[0 < c \leqslant \frac{\operatorname{Re}(Z_\varkappa - Z^0)}{|Z_\varkappa - Z^0|}\right]\right)$$

where $z^0 (Z^0)$ is the coordinate of o (O). In this case inequality (7) holds for all points $z_\varkappa$, $\varkappa = 1, 2, \ldots$

In fact, we can assume that the point o (O) is the origin and that the interior normal is the positive axis $\operatorname{Re} n > 0$ $(\operatorname{Re} N > 0)$. If we set

$$(9) \qquad \mathfrak{i} = [|n - \varrho| < \varrho] \text{ and } \mathfrak{a} = [|n + \varrho| > \varrho], \qquad \varrho > 0, \text{ sufficiently small}$$

[5] If the points $z_\varkappa \to o$, then by (2) the limit point O of $Z_\varkappa = W(z_\varkappa)$ is the image of o, i.e., $O = W(o)$.

(see Fig. 1.1), then

$$\mathfrak{j} \subset \mathfrak{d} \subset \mathfrak{a} \tag{10}$$

and the point o will simultaneously belong to the boundaries of $\mathfrak{j}$, $\mathfrak{d}$ and $\mathfrak{a}$.

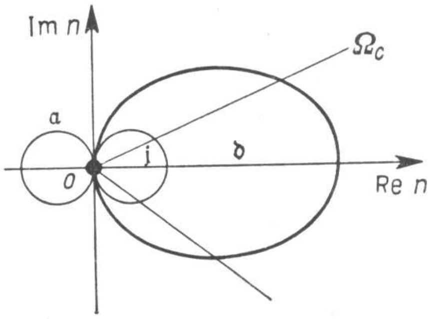

Fig. 1.1

We have

$$K_{\mathfrak{j}}(n, \bar{n}) = \frac{1}{\pi(n+\bar{n}-|n|^2/\varrho)^2}, \qquad K_{\mathfrak{a}}(n, \bar{n}) = \frac{1}{\pi(n+\bar{n}+|n|^2/\varrho)^2} \tag{11}$$

see [8], p. 37-38. From (10) it follows that

$$K_{\mathfrak{j}}(n, \bar{n}) \geqslant K_{\mathfrak{d}}(n, \bar{n}) \geqslant K_{\mathfrak{a}}(n, \bar{n}). \tag{12}$$

Therefore,

$$\frac{1}{\pi(n+\bar{n})^2\left(1-|n|^2/\varrho(n+\bar{n})\right)^2} \geqslant K_{\mathfrak{d}}(n, \bar{n}) \geqslant \frac{1}{\pi(n+\bar{n})^2\left(1+|n|^2/\varrho(n+\bar{n})\right)^2}. \tag{13}$$

Repeating the same consideration for $\mathfrak{D}$, we get

$$\frac{1}{\pi(N+\bar{N})^2\left(1-|N|^2/\varrho(N+\bar{N})\right)^2} \geqslant K_{\mathfrak{D}}(N, \bar{N}) \geqslant \frac{1}{\pi(N+\bar{N})^2\left(1+|N|^2/\varrho(N+\bar{N})\right)^2}. \tag{14}$$

Thus

$$\frac{(N+\bar{N})^2}{(n+\bar{n})^2}\,\frac{\left(1+|N|^2/\varrho(N+\bar{N})\right)^2}{\left(1-|n|^2/\varrho(n+\bar{n})\right)^2} \geqslant \frac{K_{\mathfrak{d}}(n, \bar{n})}{K_{\mathfrak{D}}(N, \bar{N})} \geqslant \frac{(N+\bar{N})^2\left(1-|N|^2/\varrho(N+\bar{N})\right)^2}{(n+\bar{n})^2\left(1+|n|^2/\varrho(n+\bar{n})\right)^2}. \tag{15}$$

If the set of points $\{n_\varkappa\}$ ($\{N_\varkappa\}$), $\varkappa = 1, 2, \ldots$, lies in the angular domain Ω_c, see (8), then

$$2c|n_\varkappa| \leqslant (n_\varkappa+\bar{n}_\varkappa) \leqslant 2|n_\varkappa| \qquad (2c|N_\varkappa| \leqslant (N_\varkappa+\bar{N}_\varkappa) \leqslant 2|N_\varkappa|).$$

From (15) it follows that

$$\text{(8a)} \qquad \frac{|N|^2(1+\varepsilon)^2}{c^2|n|^2(1-\varepsilon)^2} \geqslant \frac{K_{\mathfrak{d}}(n,\bar{n})}{K_{\mathfrak{D}}(N,\bar{N})} \geqslant \frac{c^2|N|^2(1-\varepsilon)^2}{|n|^2(1+\varepsilon)^2}.$$

$|n|$ is the distance of the point n from o and $|N|$ is the distance of N from O. ε is a positive quantity which goes to 0, for $n \to o$ $(N \to O)$.

By (2)

$$\text{(16)} \qquad \frac{|n|}{e} \leqslant |N| \leqslant e|n|.$$

Thus

$$\text{(17)} \qquad \frac{e^2(1+\varepsilon)^2}{c^2(1-\varepsilon)^2} \geqslant \frac{K_{\mathfrak{d}}(n,\bar{n})}{K_{\mathfrak{D}}(N,\bar{N})} \geqslant \frac{c^2(1-\varepsilon)^2}{e^2(1+\varepsilon)^2}$$

follows. (17) represents the desired bounds for $[K_{\mathfrak{d}}(n,\bar{n})/K_{\mathfrak{D}}(N,\bar{N})]$ in the neighborhood of the boundary point o.

We proceed now to the formulation of sufficient conditions for W, $\mathfrak{d}$ and $\mathfrak{D}$, insuring that (2) implies (3) for all points $z \in \mathfrak{d} - \partial\mathfrak{d}$.

We assume that at every boundary point $o \in \partial\mathfrak{d}$ $(O \in \partial\mathfrak{D})$ exists a normal and two circles of radius ϱ, $\varrho > 0$, with the center at the interior and exterior normal, respectively. The first circle lies inside of $\mathfrak{d}$ $(\mathfrak{D})$, the second outside of $\mathfrak{d}$ $(\mathfrak{D})$. Here ϱ is independent of the choice of the boundary point.)

From (2) it follows that the domain Ω_c in $\mathfrak{d}$ goes into a domain lying in $\Omega_{c'}$, $0 < c' < \infty$. In this way one shows that in the case of a transformation W (in the case of two real variables) (2) implies (3).

The method indicated here can be generalized to the case of QPCT's W in the case of a real four-dimensional space. However, some additional hypotheses about $\mathfrak{d}$, $\mathfrak{D} = W(\mathfrak{d})$ and the vector field (du_1, du_2), $\big((dU_1, dU_2)\big)$ as well as some additional considerations, are needed. The derivation of the results uses the classification of the boundary points in the theory of functions of two complex variables. See [2], [5], [6], [7], p. 7 ff. and 26.

In the present paper we discuss the above mentioned problem for the case of QPCT's W in the space of four real variables.

2. Bounds for the distortion of the invariant length of a vector at an interior point of $\mathfrak{d}$. The derivation of our results is based on the use of the well-known inequalities for the kernel function K and the invariant metric. Suppose $\mathfrak{a} \supset \mathfrak{d} \supset \mathfrak{j}$. Then by (30), p. 18 of [7],

$$\text{(1)} \qquad K_{\mathfrak{j}}(z,\bar{z}) \geqslant K_{\mathfrak{d}}(z,\bar{z}) \geqslant K_{\mathfrak{a}}(z,\bar{z}), \qquad (z,\bar{z}) = (z_1, z_2; \bar{z}_1, \bar{z}_2),$$

and by the formula line 8, p. 53, [6], p. 54 of [5], and (33), p. 141 of [8],

$$(2)\quad \frac{K_{\mathfrak{j}}(z,\bar{z})}{K_{\mathfrak{a}}(z,\bar{z})} ds^2_{\mathfrak{j}}(z,\bar{z},du,d\bar{u}) \geqslant ds^2_{\mathfrak{d}}(z,\bar{z};du,d\bar{u}) \geqslant \frac{K_{\mathfrak{a}}(z,\bar{z})}{K_{\mathfrak{j}}(z,\bar{z})} ds^2_{\mathfrak{a}}(z,\bar{z},du,d\bar{u}),$$

$$ds^2(z,\bar{z},du,d\bar{u}) = \sum_{n=0}^{2}\sum_{m=0}^{2} T_{m\bar{n}}(z,\bar{z})\,du_m d\bar{u}_n, \qquad du = (du_1, du_2).$$

We make the assumption that the domain $\mathfrak{d}$ and $\mathfrak{D} = W(\mathfrak{d})$ are bounded. Let us consider in $\mathfrak{d}$ a point c which has the distance $\varrho^{(1)}$, $\varrho^{(1)} > 0$, from the boundary.

Thus we use as an interior domain of comparison $\mathfrak{j}$ the hypersphere of radius $\varrho^{(1)}$ with the center at c, and as an exterior domain of comparison the hypersphere with the center at c but with radius $\varrho^{(2)}$, where $\varrho^{(2)} < \infty$ is sufficiently large. By (5a), p. 22 of [5] and (34), p. 141 of [8] if

$$\mathfrak{h} = [|z_1|^2 + |z_2|^2 < \varrho^2],$$

then

$$(3)\qquad K_{\mathfrak{h}}(z,\bar{z}) = \frac{2\varrho^2}{\pi^2(\varrho^2 - z_1\bar{z}_1 - z_2\bar{z}_2)^3}$$

and

$$(4)\qquad \begin{aligned} T_{1\bar{1}}(z,\bar{z}) &= \frac{3(\varrho^2 - z_2\bar{z}_2)}{(\varrho^2 - z_1\bar{z}_1 - z_2\bar{z}_2)^2}, \qquad T_{1\bar{2}}(z,\bar{z}) = \frac{3\bar{z}_1 z_2}{(1 - z_1\bar{z}_1 - z_2\bar{z}_2)^2}, \\ T_{2\bar{2}}(z,\bar{z}) &= \frac{3(\varrho^2 - z_1\bar{z}_1)}{(\varrho^2 - z_1\bar{z}_1 - z_2\bar{z}_2)^2}. \end{aligned}$$

If we choose the point c as the origin, $\mathfrak{j} = (|z_1|^2 + |z_2|^2 < \varrho^{(1)^2})$ and $\mathfrak{a} = [|z_1|^2 + |z_2|^2 < \varrho^{(2)^2}]$ as interior and exterior domains of comparison, respectively, then from (1)-(4) it follows that at the point $c = z \equiv (z_1, z_2)$

$$(5)\qquad \frac{2}{\pi^2 \varrho^{(1)^4}} \geqslant K_{\mathfrak{d}}(z,\bar{z}) \geqslant \frac{2}{\pi^2 \varrho^{(2)^4}},$$

$$(6)\quad 3\left(\frac{\varrho^{(2)}}{\varrho^{(1)}}\right)^4 \left[\frac{|du_1|^2}{\varrho^{(1)^2}} + \frac{|du_2|^2}{\varrho^{(1)^2}}\right] \geqslant ds^2_{\mathfrak{d}}(z,\bar{z},du,d\bar{u}) \geqslant 3\left(\frac{\varrho^{(1)}}{\varrho^{(2)}}\right)^4 \left[\frac{|du_1|^2}{\varrho^{(2)^2}} + \frac{|du_2|^2}{\varrho^{(2)^2}}\right].$$

Let us assume that the point c lies on the interior normal to $\partial\mathfrak{d}$ at the boundary point o. If $C = W(c)$, then according to (1.2) the (Euclidean) distance δ_C of C from $\partial\mathfrak{D}$ satisfies the inequality

$$(7)\qquad P^{(1)} = \delta_C \leqslant e\varrho^{(1)}.$$

Analogously, if it is possible to draw the hypersphere with the center at C of radius $P^{(1)}$ around the point C, then

$$\delta_c = \varrho^{(1)} \leqslant eP^{(1)}. \tag{8}$$

Therefore,

$$\frac{P^{(1)}}{e} \leqslant \varrho^{(1)} \leqslant eP^{(1)}. \tag{9}$$

If we denote by $\varrho^{(2)}$ $(P^{(2)})$ the maximum of the distances of the points of $\bar{\mathfrak{d}}$ $(\bar{\mathfrak{D}})$ from the boundary, then the hypersphere

$$|z_1-z_1^{(c)}|^2+|z_2-z_2^{(c)}| < \varrho^{(2)^2} \qquad (|Z_1-Z_1^{(C)}|^2+|Z_2-Z_2^{(C)}|^2 < P^{(2)^2}) \tag{10}$$

can be used as an exterior domain of comparison for $\mathfrak{d}$ $(\mathfrak{D})$. Here $c = \{z_1^{(c)}, z_2^{(c)}\}$ $(C = \{Z_1^{(C)}, Z_2^{(C)}\})$. By (2), (4) and (9) it holds

$$3\frac{e^6P^{(2)^4}}{\varrho^{(1)^6}}[|dU_1|^2+|dU_2|^2] \geqslant 3\frac{P^{(2)^4}}{P^{(1)^6}}[|dU_1|^2+|dU_2|^2] \tag{11}$$

$$\geqslant ds_{\mathfrak{D}}^2(Z, \bar{Z}, dU, d\bar{U}) \geqslant 3\frac{P^{(1)^4}}{P^{(2)^6}}[|dU_1|^2+|dU_2|^2] \geqslant 3\frac{\varrho^{(1)^4}}{e^4P^{(2)^6}}[|dU_1|^2+|dU_2|^2].$$

From (6) and (11) it follows that

$$\frac{e^4\varrho^{(2)^4}P^{(2)^6}}{\varrho^{(1)^{10}}}\left[\frac{|du_1|^2+|du_2|^2}{|dU_1|^2+|dU_2|^2}\right] \geqslant \frac{ds_{\mathfrak{d}}^2(z, \bar{z}, du, d\bar{u})}{ds_{\mathfrak{D}}^2(Z, \bar{Z}, dU, d\bar{U})} \tag{12}$$

$$\geqslant \frac{\varrho^{(1)^{10}}}{e^6\varrho^{(2)^6}P^{(2)^4}}\left[\frac{|du_1|^2+du_2|^2}{|dU_1|^2+dU_2|^2}\right].$$

By (1.2)

$$e^2 \geqslant \frac{|du_1|^2+|du_2|^2}{|dU_1|^2+|dU_2|^2} \geqslant \frac{1}{e^2} > 0. \tag{13}$$

(12) and (13) yield bounds

$$\frac{e^6\varrho^{(2)^4}P^{(2)^6}}{\varrho^{(1)^{10}}} \geqslant \frac{ds_{\mathfrak{d}}^2(z, \bar{z}, du, d\bar{u})}{ds_{\mathfrak{D}}^2(Z, \bar{Z}, dU, d\bar{U})} \geqslant \frac{\varrho^{(1)^{10}}}{e^8\varrho^{(2)^6}P^{(2)^4}}. \tag{14}$$

Thus we see that for points c lying in a subdomain $\bar{\mathfrak{d}}_1$ inside of $\mathfrak{d}$ we obtain in (14) the bounds for the ratio $(ds_{\mathfrak{d}}^2/ds_{\mathfrak{D}}^2)$. Consequently, to obtain the desired results it is sufficient to derive corresponding bounds for sets $\{z^{(\nu)}\}$, $\nu = 1, 2, \ldots,$ of points which converge to a boundary point of $\mathfrak{d}$. In §§ 3-5 of this paper and in [10a] we discuss this problem.

3. Bounds for $ds_{\mathfrak{d}}/ds_{\mathfrak{D}}$ in the neighborhood of points of the third order. In the present section we consider the distortion of the non-Euclidean length of a vector at a point c, which lies in a neighborhood of a boundary point of the third order.

We assume:

1. The points c and $C = W(c)$ lie in a neighborhood of the boundary points o and $O = W(o)$, respectively. o and O are the points of the third order. See [7], p. 12 and 20, and [2].

2. The boundary $\mathfrak{b}^3$ ($\mathfrak{B}^3 = W(\mathfrak{b}^3)$) of $\mathfrak{d}$ ($\mathfrak{D} = W(\mathfrak{d})$) has at the point o (O) the tangential analytic plane, which lies outside $\mathfrak{d}-o$ ($\mathfrak{D}-O$). Let $n_1 = 0$ ($N_1 = 0$) be the equation of the analytic tangential plane at o (O) and $n_2 = 0$ ($N_2 = 0$) of the perpendicular plane. n_1, n_2 (N_1, N_2) are called *coordinates normal at the point* o (O).

3. We assume that

$$\mathfrak{i} = [n_1+\bar{n}_1-\frac{1}{\varrho^{(1)}}(|n_1|^2+|n_2|^2) > 0] \tag{1}$$

and

$$\mathfrak{a} = [n_1+\bar{n}_1-\frac{1}{\varrho^{(2)}}(|n_1|^2+|n_2|^2) > 0], \qquad \varrho^{(2)} > \varrho^{(1)} > 0, \tag{2}$$

are interior and exterior domains of comparison for $\mathfrak{d}$ at o, respectively.

4. Analogously we assume that

$$\mathfrak{J} = [N_1+\bar{N}_1-\frac{1}{P^{(1)}}(|N_1|^2+|N_2|^2) > 0] \tag{3}$$

and

$$\mathfrak{A} = [N_1+\bar{N}_1-\frac{1}{P^{(2)}}(|N_1|^2+|N_2|^2) > 0], \qquad P^{(2)} > P^{(1)} > 0, \tag{4}$$

are interior and exterior domains of comparison for $\mathfrak{D}$ at O, respectively.

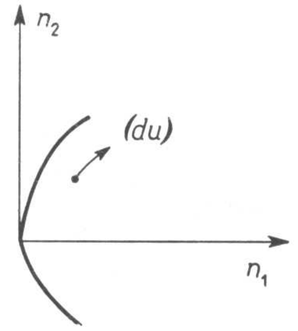

Fig. 3.1

5. The point (n_1, n_2) $((N_1, N_2))$ lies in the cone

$$\Omega_c = \left[0 < \frac{1}{c_2} < \frac{(n_1+\bar{n}_1)}{2(|n_1|^2+|n_2|^2)^{1/2}}\right], \tag{5}$$

$$\left(\Omega_C = \left[0 < \frac{1}{c_2} < \frac{(N_1+\bar{N}_1)}{2(|N_1|^2+|N_2|^2)^{1/2}}\right]\right), \qquad c_2 > 2.$$

6. The vectors (du_1, du_2) and $(dU_1, dU_2) = W(du_1, du_2)$ satisfy the condition C, namely,

$$0 < c_1^2 \leqslant \frac{|du_1|^2}{|du_1|^2+|du_2|^2} \leqslant c_5^2, \qquad 0 < c_5 \leqslant 1,\ c_1 < c_5, \tag{6}$$

or

$$\left(\frac{1}{c_1^2} - 1\right)|du_1|^2 \geqslant |du_2|^2 \geqslant \left(\frac{1}{c_5^2} - 1\right)|du_1|^2,$$

$$0 < c_1^2 \leqslant \frac{|dU_1|^2}{|dU_1|^2+|dU_2|^2} \leqslant c_5^2. \tag{7}$$

THEOREM 3.1. *Let W be a* QPCT *mapping the domain $\mathfrak{d}$ in z_1, z_2-space onto the domain $\mathfrak{D}$ in Z_1, Z_2-space. $z_k = x_k+iy_k$, $Z_k = X_k+iY_k$. W is continuous, in particular, if $L(p_1, p_2)$ is the distance between two points $p_1, p_2 \in \mathfrak{d}$, and $L(P_1, P_2)$ is the distance between their images $P_k = W(p_k)$ in $\mathfrak{D}$, there exists a constant e, such that (1.2) holds. Further we assume that the hypotheses 1-6 hold and the constants ϱ^ν (P^ν), c_k are independent of the choice of boundary point of $\mathfrak{d}$ $(\mathfrak{D})$.*

Let (du_1, du_2) be a vector in $\mathfrak{d}$ at the point $c = (n_1, n_2)$, its image $(dU_1, dU_2) = W(du_1, du_2)$ lies at $C \in \mathfrak{D}$. The above vectors satisfy the conditions (6) and (7), respectively. If $s_{\mathfrak{d}}(n, \bar{n}; du, d\bar{u})$ $\big(s_{\mathfrak{D}}(N, \bar{N};\, dU, d\bar{U})\big)$ is the non-Euclidean length of (du_1, du_2) $\big((dU_1, dU_2)\big)$ with respect to the domain $\mathfrak{d}$ $(\mathfrak{D})$, then there exists a constant c_3, such that

$$0 < \frac{1}{c_3} \leqslant \frac{s_{\mathfrak{d}}(n, n; du, d\bar{u})}{s_{\mathfrak{D}}(N, \bar{N}; dU, d\bar{U})} \leqslant c_3 < \infty. \tag{8}$$

Proof. In accordance with (5.a), p. 22 of [5] (see also (4.22), p. 17 of [2])

$$K_{\mathfrak{i}}(n; \bar{n}) = \frac{2\varrho^{(1)^2}}{\pi^2[\varrho^{(1)}(n_1+\bar{n}_1) - |n_1|^2 - |n_2|^2]^3}, \tag{9}$$

$$K_{\mathfrak{Z}}(N; \bar{N}) = \frac{2P^{(1)^2}}{\pi^2[P^{(1)}(N_1+\bar{N}_1) - |N_1|^2 - |N_2|^2]^3} \tag{10}$$

and similar formulas hold for $K_{\mathfrak{a}}$ $(K_{\mathfrak{A}})$ with $\varrho^{(1)}$ $(P^{(1)})$ replaced by $\varrho^{(2)}$ $(P^{(2)})$, resp. If $c = (n_1, n_2)$ and $C = W(c) = (N_1, N_2)$, then by (2.1) and (2.2),

$$\text{(11)} \qquad \frac{2}{\pi^2 \varrho^{(1)} (n_1+\bar{n}_1)^3 \left[1 - \frac{1}{\varrho^{(1)}} \frac{n_1\bar{n}_1 + n_2\bar{n}_2}{n_1+\bar{n}_1}\right]^3} \geqslant K_{\mathfrak{b}}(n, \bar{n})$$

$$\geqslant \frac{2}{\pi^2 \varrho^{(2)} (n_1+\bar{n}_1)^3 \left[1 - \frac{1}{\varrho^{(2)}} \frac{n_1\bar{n}_1 + n_2\bar{n}_2}{n_1+\bar{n}_1}\right]^3}.$$

We note that

$$T_{11} = \frac{3({\varrho^{(1)}}^2 - n_2\bar{n}_2)}{[\varrho^{(1)}(n_1+\bar{n}_1) - n_1\bar{n}_1 - n_2\bar{n}_2]^2}, \qquad T_{1\bar{2}} = \frac{-3(\varrho^{(1)} - \bar{n}_1)n_2}{[\varrho^{(1)}(n_1+\bar{n}_1) - n_1\bar{n}_1 - n_2\bar{n}_2]^2},$$

$$T_{2\bar{2}} = \frac{3[\varrho^{(1)}(n_1+\bar{n}_1) - |n_1|^2]}{[\varrho^{(1)}(n_1+\bar{n}_1) - |n_1|^2 - |n_2|^2]^2}.$$

By (2.2),

$$\text{(12)} \qquad \tilde{h}_1(n, \bar{n}, du, d\bar{u}) \equiv \frac{3\varrho^{(2)}\left[1 - \frac{1}{\varrho^{(2)}} \frac{n_1\bar{n}_1 + n_2\bar{n}_2}{n_1+\bar{n}_1}\right]^3}{{\varrho^{(1)}}^3 (n_1+\bar{n}_1)^2 \left[1 - \frac{1}{\varrho^{(1)}} \frac{n_1\bar{n}_1 + n_2\bar{n}_2}{n_1+\bar{n}_1}\right]^5} \times$$

$$\times\{({\varrho^{(1)}}^2 - n_2\bar{n}_2)|du_1|^2 - (\varrho^{(1)} - \bar{n}_1)n_2 du_1 d\bar{u}_2 -$$

$$- (\varrho^{(1)} - n_1)\bar{n}_2 d\bar{u}_1 du_2 + \big(\varrho^{(1)}(n_1+\bar{n}_1) - n_1\bar{n}_1\big)|du_2|^2\}$$

$$\geqslant ds^2_{\mathfrak{b}}(n_1, n_2, du_1, du_2)$$

$$\geqslant \frac{3\varrho^{(1)}\left[1 - \frac{1}{\varrho^{(1)}} \frac{n_1\bar{n}_1 + n_2\bar{n}_2}{n_1+\bar{n}_1}\right]^3}{{\varrho^{(2)}}^3 (n_1+\bar{n}_1)^2 \left[1 - \frac{1}{\varrho^{(2)}} \frac{n_1\bar{n}_1 + n_2\bar{n}_2}{n_1+\bar{n}_1}\right]^5} \times$$

$$\times\{({\varrho^{(2)}}^2 - |n_2|^2)|du_1|^2 - (\varrho^{(2)} - \bar{n}_1)n_2 du_1 d\bar{u}_2 -$$

$$- (\varrho^{(2)} - n_1)\bar{n}_2 d\bar{u}_1 du_2 + (\varrho^{(2)}(n_1+\bar{n}_1) - |n_1|^2)|du_2|^2\}$$

$$\equiv \tilde{h}_2(n, \bar{n}, du, d\bar{u}).$$

Since, by 5 and 6,

$$\text{(13)} \qquad \frac{1}{4}(n_1+\bar{n}_1)^2 \leqslant |n_1|^2 \leqslant |n_1|^2 + |n_2|^2 \leqslant \frac{c_2^2(n_1+\bar{n}_1)^2}{4},$$

$$\text{(14)} \qquad {\varrho^{(1)}}^2 - c_2^2(n_1+\bar{n}_1)^2 \leqslant {\varrho^{(1)}}^2 - |n_2|^2 \leqslant {\varrho^{(1)}}^2,$$

$$\text{(15)} \qquad c_1^2 \leqslant \frac{|du_1|^2}{|du_1|^2 + |du_2|^2} \quad \text{or} \quad |du_2|^2 \leqslant \left(\frac{1}{c_1^2} - 1\right)|du_1|^2,$$

$$(16)\qquad |\varrho^{(1)}-\bar{n}_1| \leqslant \varrho^{(1)}+\sqrt{|n_1|^2+|n_2|^2} \leqslant \varrho^{(1)}+\frac{c_2}{2}(n_1+\bar{n}_1),$$

$$(17)\qquad |n_2| \leqslant (|n_1|^2+|n_2|^2)^{1/2} \leqslant \frac{c_2}{2}(n_1+\bar{n}_1),$$

$$(18)\qquad \frac{3\varrho^{(2)}\left[1-\dfrac{1}{4\varrho^{(2)}}(n_1+\bar{n}_1)\right]^3|du_1|^2}{{\varrho^{(1)}}^3(n_1+\bar{n}_1)^2\left[1-\dfrac{c_2^2}{4\varrho^{(1)}}(n_1+\bar{n}_1)\right]^5}\left\{{\varrho^{(1)}}^2+\left(\varrho^{(1)}+\frac{c_2}{2}(n_1+\bar{n}_1)\right)\times\right.$$

$$\left.\times c_2(n_1+\bar{n}_1)\left(\frac{1}{c_1^2}-1\right)^{1/2}+\varrho^{(1)}(n_1+\bar{n}_1)\left(\frac{1}{c_1^2}-1\right)\right\} \geqslant \tilde{h}_1(n_1, n_2; du_1, du_2)$$

and

$$(19)\qquad \tilde{h}_2(n, \bar{n}; du, d\bar{u}) \geqslant \frac{3\varrho^{(1)}\left[1-\dfrac{c_2^2}{4\varrho^{(1)}}(n_1+\bar{n}_1)\right]^3|du_1|^2}{{\varrho^{(2)}}^3(n_1+\bar{n}_1)^2\left[1-\dfrac{1}{4\varrho^{(2)}}(n_1+\bar{n}_1)\right]^5}\times$$

$$\times\left\{[{\varrho^{(2)}}^2-c_2^2(n_1+\bar{n}_1)^2]-c_2\left(\varrho^{(2)}+\frac{c_2}{2}(n_1+\bar{n}_1)\right)(n_1+\bar{n}_1)\left(\frac{1}{c_1^2}-1\right)^{1/2}+\right.$$

$$\left.+\varrho^{(2)}(n_1+\bar{n}_1)\left(1-\frac{c_2^2}{4\varrho^{(2)}}(n_1+\bar{n}_1)\right)(1-c_5^2)\right\}.$$

Analogous inequalities can be derived for $ds_{\mathfrak{D}}^2(N_1, N_2; dU_1, dU_2)$. Therefore by (18), (19),

$$(20)\qquad \frac{\varrho^{(2)}{P^{(2)}}^3[1-(n_1+\bar{n}_1)/4\varrho^{(2)}]^3[1-(N_1+\bar{N}_1)/4P^{(2)}]^5G_1|du_1|^2}{{\varrho^{(1)}}^3P^{(1)}[1-c_2^2(n_1+\bar{n}_1)/4\varrho^{(1)}]^5[1-c_2^2(N_1+\bar{N}_1)/4P^{(1)}]^3L_1|dU_1|^2}$$

$$\geqslant \frac{ds_{\mathfrak{d}}^2(n_1, n_2; du_1, du_2)}{ds_{\mathfrak{D}}^2(N_1, N_2; dU_1, dU_2)}$$

$$\geqslant \frac{\varrho^{(1)}{P^{(1)}}^3[1-c_2^2(n_1+\bar{n}_1)/4\varrho^{(1)}]^3[1-c_2^2(N_1+\bar{N}_1)/4P^{(1)}]^5G_2|du_1|^2}{{\varrho^{(2)}}^3P^{(2)}[1-(n_1+\bar{n}_1)/4\varrho^{(2)}]^5[1-(N_1+\bar{N}_1)/4P^{(1)}]^3L_2|dU_1|^2},$$

where

$$G_1 = \{{\varrho^{(1)}}^2+c_2(n_1+\bar{n}_1)(c_1^{-2}-1)^{1/2}(\varrho^{(1)}+\tfrac{1}{2}c_2(n_1+\bar{n}_1))+$$
$$+\varrho^{(1)}(c_1^{-2}-1)(n_1+\bar{n}_1)\}(N_1+\bar{N}_1)^2,$$

$$L_1 = \left\{{P^{(2)}}^2-c_2^2(N_1+\bar{N}_1)^2-c_2(c_1^{-2}-1)^{1/2}(N_1+\bar{N}_1)(P^{(2)}+\tfrac{1}{2}c_2(N_1+\bar{N}_1))+\right.$$

$$\left.+P^{(2)}(1-c_5^2)(N_1+\bar{N}_1)\left[1-\frac{c_2^2}{4P^{(2)}}(N_1+\bar{N}_1)\right]\right\}(n_1+\bar{n}_1)^2.$$

Note. $(1-c_5^2)$ can be replaced by $(1-c_5)$ since $1+c_5>1$.

$$G_2=\{\varrho^{(2)^2}-c_2^2(n_1+\bar{n}_1)^2-c_2(c_1^{-2}-1)^{1/2}(n_1+\bar{n}_1)\big(\varrho^{(2)}+c_2(n_1+\bar{n}_1)/2)\big)+\\ +\varrho^{(2)}(1-c_5^2)(n_1+\bar{n}_1)\big(1-c_2^2(n_1+\bar{n}_1)/4\varrho^{(2)}\big)\}(N_1+\bar{N}_1)^2,$$

$$L_2=\{P^{(1)^2}+c_2(c_1^{-2}-1)^{1/2}(N_1+\bar{N}_1)\big(P^{(1)}+c_2(N_1+\bar{N}_1)/2\big)+\\ +P^{(1)}(c_1^{-2}-1)(N_1+\bar{N}_1)\}(n_1+\bar{n}_1)^2.$$

By (1.2)

(21) $$e^2\geqslant\frac{|n_1|^2+|n_2|^2}{|N_1|^2+|N_2|^2}\geqslant\frac{1}{e^2},$$

by (5)

(22) $$4\geqslant\frac{(n_1+\bar{n}_1)^2}{|n_1|^2+|n_2|^2}>\frac{4}{c_2^2},\qquad 4\geqslant\frac{(N_1+\bar{N}_1)^2}{|N_1|^2+|N_2|^2}>\frac{4}{c_2^2}.$$

Therefore,

(23)
$$c_2^2e^2\geqslant\frac{(n_1+\bar{n}_1)^2}{(N_1+\bar{N}_1)^2}=\frac{(n_1+\bar{n}_1)^2}{4(|n_1|^2+|n_2|^2)}\,\frac{|n_1|^2+|n_2|^2}{|N_1|^2+|N_2|^2}\,\frac{4(|N_1|^2+|N_2|^2)}{(N_1+\bar{N}_1)^2}\geqslant\frac{1}{c_2^2e^2}.$$

Thus for $(n_1+\bar{n}_1)\to 0$ follows

(24) $$(N_1+\bar{N}_1)\to 0.$$

By (1.2)

(25) $$\frac{1}{e^2}\leqslant\frac{|du_1|^2+|du_2|^2}{|dU_1|^2+|dU_2|^2}\leqslant e^2<\infty.$$

From (20), (23), (24) and (25) follows that for sufficiently small $(n_1+\bar{n}_1)$ the right and the left-hand side of (20) converges to c_3 and $1/c_3$, $c_3<\infty$, respectively. This proves inequality (8).

4. Bounds for $ds_{\mathfrak{d}}/ds_{\mathfrak{D}}$ in the neighborhood of points of the fourth order. In the present and in the next sections we shall consider QPCT's W of a domain $\tilde{\mathfrak{d}}$ which is bounded by two analytic hypersurfaces $\tilde{\mathfrak{i}}_1^3$, $\tilde{\mathfrak{i}}_2^3$.

$\partial(\tilde{\mathfrak{d}})=\tilde{\mathfrak{i}}_1^3\cup\tilde{\mathfrak{i}}_2^3$; $\tilde{\mathfrak{f}}^2=\tilde{\mathfrak{i}}_1^3\cap\tilde{\mathfrak{i}}_2^3$ is the distinguished boundary of $\tilde{\mathfrak{d}}$. We assume that $\tilde{\mathfrak{d}},\tilde{\mathfrak{d}}^3=\partial(\tilde{\mathfrak{d}})$ and $\tilde{\mathfrak{f}}^2$ are homeomorphic to the bicylinder $[|\zeta_1|<1,|\zeta_2|<1]$, to $([|\zeta_1|\leqslant 1,\ |\zeta_2|=1]\cup[|\zeta_1|=1,\ |\zeta_2|\leqslant 1])$ and to the surface $[|\zeta_1|=1,|\zeta_2|=1]$, respectively. $\tilde{\mathfrak{D}}=W(\tilde{\mathfrak{d}})$, $\tilde{\mathfrak{D}}^3=\partial(\tilde{\mathfrak{D}})=W(\tilde{\mathfrak{d}}^3)$, $\tilde{\mathfrak{F}}^2=W(\tilde{\mathfrak{f}}^2)$ have the same topological structures as $\tilde{\mathfrak{d}},\tilde{\mathfrak{d}}^3$ and $\tilde{\mathfrak{f}}^2$, resp.:

(1) $$\tilde{\mathfrak{D}}^3=\tilde{\mathfrak{J}}_1^3\cup\tilde{\mathfrak{J}}_2^3,\qquad \tilde{\mathfrak{J}}_k^3=W(\tilde{\mathfrak{i}}_k^3),$$

where $\tilde{\mathfrak{J}}_k^3$, $k=1,2,\ldots$, are segments of analytic hypersurfaces. In this section we shall consider a point $\tilde{o}$ $(\tilde{O})$ of $\tilde{\mathfrak{f}}^2$ $(\tilde{\mathfrak{F}}^2)$.

We assume that

$$\tilde{\mathfrak{i}}_k^3 = \bigcup_{\lambda_k} \tilde{\mathfrak{i}}_k^2(\lambda_k), \qquad k = 1, 2,$$

where $\tilde{\mathfrak{i}}_k^2(\lambda_k)$ are segments of analytic surfaces $\big(\tilde{\mathfrak{J}}_k^3 = \bigcup_{\lambda_k} \tilde{\mathfrak{J}}_k^2(\lambda_k)\big)$.

Let

$$\tilde{o} = \overline{\tilde{\mathfrak{i}}_1^2(\lambda_1^0)} \frown \overline{\tilde{\mathfrak{i}}_2^2(\lambda_2^0)} \qquad \big(\tilde{O} = \overline{\tilde{\mathfrak{J}}_1^2(\lambda_1^0)} \frown \overline{\tilde{\mathfrak{J}}_2^2(\lambda_2^0)}\big) \tag{2}$$

where $\mathfrak{i}_k^2(\lambda_k^0)$ $\big(\mathfrak{J}_k^2(\lambda_k^0)\big)$, $k = 1, 2$, is a segment of the surface $\tilde{z}_1 = h_1(\tilde{z}_2, \lambda_1^0)$ and $\tilde{z}_2 = h_2(\tilde{z}_1, \lambda_2^0)$ $\big(\tilde{Z}_1 = H_1(\tilde{Z}_2, \lambda_1^0)$ and $\tilde{Z}_2 = H_2(\tilde{Z}_1, \lambda_2^0)\big)$, respectively. Here $h_k(\tilde{z}_{3-k}, \lambda_k^0)$ $\big(H_k(\tilde{Z}_{3-k}, \lambda_k^0)\big)$ are analytic functions of $\tilde{z}_{3-k}$ $(\tilde{Z}_{3-k})$ which are regular in a sufficiently large domain. We assume that the mapping

$$t = [z_1 = \tilde{z}_1 - h_1(\tilde{z}_2, \lambda_1^0),\ z_2 = \tilde{z}_2 - h_2(\tilde{z}_1, \lambda_2^0)] \tag{3}$$

is one-to-one and analytic in $\overline{\tilde{\mathfrak{d}}}$. Analogously,

$$T = [Z_1 = \tilde{Z}_1 - H_1(\tilde{Z}_2, \lambda_1^0),\ Z_2 = \tilde{Z}_2 - H_2(\tilde{Z}_1, \lambda_2^0)] \tag{4}$$

is assumed to be one-to-one and analytic in $\overline{\tilde{\mathfrak{D}}}$.

By (3) and (4) the domain $\tilde{\mathfrak{d}}$ $(\tilde{\mathfrak{D}})$ will be transformed into the domain $\mathfrak{d}$ $(\mathfrak{D})$, the segments $\tilde{\mathfrak{i}}_k^2(\lambda_k^0)$ $\big(\tilde{\mathfrak{J}}_k^2(\lambda_k^0)\big)$, $k = 1, 2$, into the segments $\mathfrak{i}_k^2(\lambda_k^0)$ of $z_1 = 0$ and $z_2 = 0$ $(\mathfrak{J}_k^2(\lambda_k^0)$ of $Z_1 = 0$ and $Z_2 = 0)$, respectively. Since (3) and (4) are PCT's, the invariant lengths remain unchanged. Instead of the domains $\tilde{\mathfrak{d}}$ and $\tilde{\mathfrak{D}}$ we shall consider the domains $\mathfrak{d}$ and $\mathfrak{D}$, respectively.

The boundary $\mathfrak{d}^3$ has at the point o two tangential analytic planes. The coordinate system n_1, n_2, where $n_1 = 0$ and $n_2 = 0$ are these tangential planes, is called normal with respect to the point o.

In an analogous manner we introduce in $\mathfrak{D}$ coordinates N_1, N_2 which are normal with respect to the boundary point O of $\overline{\mathfrak{D}}$. Obviously, $o \in \mathfrak{f}^2$ $(O \in \mathfrak{F}^2)$.

3a. We make the assumption that

$$\tilde{\mathfrak{j}} \subset \tilde{\mathfrak{d}} \subset \tilde{\mathfrak{a}} \tag{5}$$

where $\tilde{\mathfrak{j}}$ and $\tilde{\mathfrak{a}}$ are domains which one obtains by PCT t^{-1} (inverse to t) from

$$\mathfrak{j} = [\varrho^{(1)}(n_1 + \overline{n}_1) - |n_1|^2 > 0,\ \varrho^{(1)}(n_2 + \overline{n}_2) - |n_2|^2 > 0], \qquad \varrho^{(1)} > 0, \tag{6}$$

and

$$\mathfrak{a} = [\varrho^{(2)}(n_1 + \overline{n}_1) + |n_1|^2 > 0,\ \varrho^{(2)}(n_2 + \overline{n}_2) + |n_2|^2 > 0], \qquad \varrho^{(2)} > 0 \tag{7}$$

respectively.

4a. Analogously, we make the assumption that

$$\tilde{\mathfrak{J}} \subset \tilde{\mathfrak{D}} \subset \tilde{\mathfrak{A}}, \tag{8}$$

where $\tilde{\mathfrak{J}}$ and $\tilde{\mathfrak{A}}$ are domains which one obtains by PCT T^{-1} (inverse to T) from

$$\mathfrak{J} = [P^{(1)}(N_1+\overline{N}_1) - |N_1|^2 > 0,\ P^{(1)}(N_2+\overline{N}_2) - |N_2|^2 > 0], \qquad P^{(1)} > 0, \tag{9}$$

and

$$\mathfrak{A} = [P^{(2)}(N_1+\overline{N}_1) + |N_1|^2 > 0,\ P^{(2)}(N_2+\overline{N}_2) + |N_2|^2 > 0], \qquad P^{(2)} > 0. \tag{10}$$

5a. The point $c = (n_1, n_2)$ $(C = (N_1, N_2))$ lies in the cone (3.5)

6a. The vectors (du_1, du_2) $((dU_1, dU_2))$ satisfy conditions (3.6) and (3.7).

In addition to the hypotheses 3a, 4a, 5a and 6a we shall assume that the point $c = (n_1, n_2)$ $(C = (N_1, N_2))$ lies in the domain

$$0 < \frac{1}{c_4} \leqslant \frac{n_k+\overline{n}_k}{2\,|n_k|} \leqslant 1 \qquad \left(\frac{1}{c_4} \leqslant \frac{N_k+\overline{N}_k}{2\,|N_k|} \leqslant 1\right), \qquad k = 1, 2. \tag{11}$$

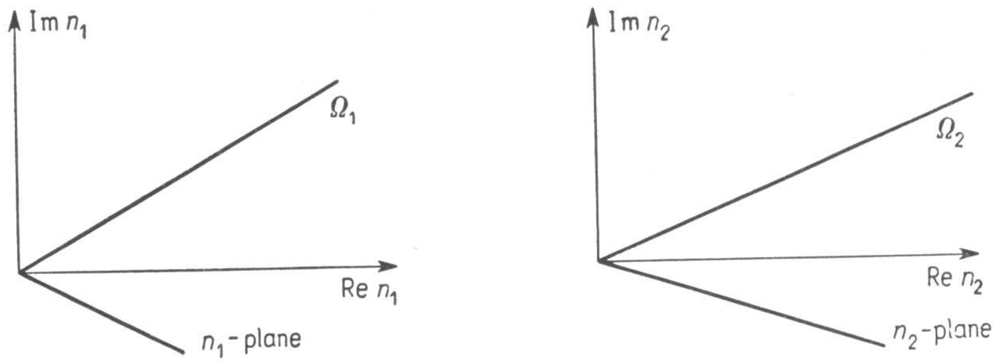

Fig. 4.1

Further we assume that

$$\begin{aligned} &c_7(n_1+\overline{n}_1) \leqslant (n_2+\overline{n}_2) \leqslant c_6(n_1+\overline{n}_1), \qquad c_6 \geqslant 1, \\ &\big(c_7(N_1+\overline{N}_1) \leqslant (N_2+\overline{N}_2) \leqslant c_6(N_1+\overline{N}_1)\big), \qquad c_7 \leqslant 1. \end{aligned} \tag{12}$$

Remark. It should be noted that condition (12) can be replaced by

$$c_7^*\,|n_1| \leqslant |n_2| \leqslant c_6^*\,|n_1|. \tag{12*}$$

From (11) follows that

$$|n_k| \geqslant \frac{n_k+\overline{n}_k}{2} \geqslant \frac{|n_k|}{c_4}. \tag{12**}$$

Therefore from (12^*) it follows $c_7^*(n_1+\overline{n}_1)/c_4 \leqslant n_2+\overline{n}_2$ and $n_2+\overline{n}_2 \leqslant c_6^* c_4(n_1+\overline{n}_1)$.

(12^*) means that the point (n_1, n_2) approaches to the point o in the product of two angular domains.

THEOREM 4.1. *Let W be a QPCT of the domain $\tilde{\mathfrak{d}}$ onto $\tilde{\mathfrak{D}}$. $\tilde{\mathfrak{d}}$ is bounded by two segments of analytic hypersurfaces $\tilde{\mathfrak{i}}_k^3$, $k = 1, 2$, and possesses the distinguished surface $\tilde{\mathfrak{f}}^2$; $\tilde{\mathfrak{D}} = W(\tilde{\mathfrak{d}})$ is again a domain bounded by two segments of analytic hypersurfaces $\tilde{\mathfrak{J}}_k^3 = W(\tilde{\mathfrak{i}}_k^3)$ and possesses the distinguished boundary $\tilde{\mathfrak{F}}^2 = W(\tilde{\mathfrak{f}}^2)$. We assume that relation (1.2) holds for $L(p_1, p_2)/L(P_1, P_2)$, $P_k = W(p_k)$. Let $W_1 = TWt^{-1}$ and let the hypotheses on p. 79 and 80 hold.*

Let $o \in \mathfrak{f}^2$ and $O \in \mathfrak{F}^2$. Further let W_1 transform the vector (du_1, du_2) at the point c into the vector (dU_1, dU_2) at the point $C = W_1(c)$, and we assume that c lies in a neighborhood of o (C in a neighborhood of O).

If $s_{\mathfrak{d}}(n, \overline{n}; du, d\overline{u})$ is the non-Euclidean length (6) *with respect to $\mathfrak{d}$ of the vector (du_1, du_2) and $s_{\mathfrak{D}}(N, \overline{N}; dU, d\overline{U})$ with respect to $\mathfrak{D}$ of the transformed vector, the ratios of $s_{\mathfrak{d}}$ and $s_{\mathfrak{D}}$ satisfy inequality* (3.8), *in this case c_3 is given in* (24).

Proof. By (4), p. 21 of [5] it holds

$$K_{\mathfrak{i}}(n, \overline{n}) = \frac{\varrho^{(1)^4}}{\pi^2 \prod_{k=1}^{2} [\varrho^{(1)}(n_k+\overline{n}_k) - |n_k|^2]^2} \tag{13}$$

and a similar expression for $K_{\mathfrak{a}}(n, \overline{n})$, with $\varrho^{(1)}$ replaced by $\varrho^{(2)}$. Analogous expressions hold for $K_{\mathfrak{J}}(N, \overline{N})$ and $K_{\mathfrak{A}}(N, \overline{N})$, where $\varrho^{(k)}$ has to be replaced by $P^{(k)}$, $k = 1, 2$.

By (7) and (2.1)

$$\frac{1}{\pi^2 \prod_{k=1}^{2} (n_k+\overline{n}_k)^2 \left[1 - \dfrac{|n_k|^2}{\varrho^{(1)}(n_k+\overline{n}_k)}\right]^2} \tag{14}$$

$$\geqslant K_{\mathfrak{d}}(n, \overline{n}) \geqslant \frac{1}{\pi^2 \prod_{k=1}^{2} (n_k+\overline{n}_k)^2 \left[1 + \dfrac{|n_k|^2}{\varrho^{(2)}(n_k+\overline{n}_k)}\right]^2}$$

(6) Since the mappings t and T are PCT's, the non-Euclidean lengths of the vectors are preserved in these transformations when passing from $\tilde{\mathfrak{d}}$ to $\mathfrak{d}$ ($\tilde{\mathfrak{D}}$ to $\mathfrak{D}$). The PCT t (T) preserves relation (1.2) but the value of e can change. See Lemma 2, p. 34 of [10a].

and by (2.2)

(15) $\tilde{h}_3(n, \bar{n}, du, d\bar{u})$

$$= \frac{2\prod_{k=1}^{2}\left[1+\frac{|n_k|^2}{\varrho^{(2)}(n_k+\bar{n}_k)}\right]^2}{\prod_{k=1}^{2}\left[1-\frac{|n_k|^2}{\varrho^{(1)}(n_k+\bar{n}_k)}\right]^2}\left[\sum_{k=1}^{2}\frac{|du_k|^2}{(n_k+\bar{n}_k)^2\left[1-\frac{|n_k|^2}{\varrho^{(1)}(n_k+\bar{n}_k)}\right]^2}\right]$$

$$\geqslant ds_{\mathfrak{d}}^2(n, \bar{n}, du, d\bar{u}) \geqslant \frac{2\prod_{k=1}^{2}\left[1-\frac{|n_k|^2}{\varrho^{(1)}(n_k+\bar{n}_k)}\right]^2}{\prod_{k=1}^{2}\left[1+\frac{|n_k|^2}{\varrho^{(2)}(n_k+\bar{n}_k)}\right]^2}\times$$

$$\times\left[\sum_{k=1}^{2}\frac{|du_k|^2}{(n_k+\bar{n}_k)^2\left[1+\frac{|n_k|^2}{\varrho^{(2)}(n_k+\bar{n}_k)}\right]^2}\right] \equiv \tilde{h}_4(n, \bar{n}, du, d\bar{u}).$$

Similar inequalities hold for $ds_{\mathfrak{D}}^2(N, \bar{N}, dU, d\bar{U})$.
From (11) it follows that

$$\frac{n_k+\bar{n}_k}{4} \leqslant \frac{|n_k|^2}{n_k+\bar{n}_k} \leqslant \frac{c_4^2}{4}(n_k+\bar{n}_k). \tag{16}$$

(We note that $\frac{1}{2}(n_k+\bar{n}_k) = \mathrm{Re}(n_k) > 0$.) Therefore by (12)

(17)

$$\frac{2\prod_{k=1}^{2}\left[1+\frac{c_4^2(n_k+\bar{n}_k)}{4\varrho^{(2)}}\right]^2[|du_1|^2+|du_2|^2]}{\prod_{k=1}^{2}\left[1-\frac{c_4^2(n_k+\bar{n}_k)}{4\varrho^{(1)}}\right]^2 c_7^2(n_1+\bar{n}_1)^2\left[1-\frac{c_4^2c_6(n_1+\bar{n}_1)}{4\varrho^{(1)}}\right]^2} \geqslant \tilde{h}_3(n, \bar{n}, du, d\bar{u}),$$

(18)

$$\tilde{h}_4(n, \bar{n}, du, d\bar{u}) \geqslant \frac{2\prod_{k=1}^{2}\left[1-\frac{c_4^2(n_k+\bar{n}_k)}{4\varrho^{(1)}}\right]^2[|du_1|^2+|du_2|^2]}{\prod_{k=1}^{2}\left[1+\frac{c_4^2(n_k+\bar{n}_k)}{4\varrho^{(2)}}\right]^2 c_6^2(n_1+\bar{n}_1)^2\left[1+\frac{c_4^2c_6(n_1+\bar{n}_1)}{4\varrho^{(2)}}\right]^2}.$$

One obtains similar bounds for $\tilde{H}_3(N, \bar{N}, dU, d\bar{U})$ and $\tilde{H}_4(N, \bar{N}, dU, d\bar{U})$. Thus

$$\frac{G_3[|du_1|^2+|du_2|^2]}{L_3[|dU_1|^2+|dU_2|^2]} \geqslant \frac{ds_{\mathfrak{d}}^2(n, \bar{n}, du, d\bar{u})}{ds_{\mathfrak{D}}^2(N, \bar{N}, dU, d\bar{U})} \geqslant \frac{G_4[|du_1|^2+|du_2|^2]}{L_4[|dU_1|^2+|dU_2|^2]}, \tag{19}$$

where

$$G_3 = c_6^2(N_1+\overline{N}_1)^2 \prod_{k=1}^{2}\left[\left(1+\frac{c_4^2(n_k+\overline{n}_k)}{4\varrho^{(2)}}\right)^2 \times \right.$$

$$\left. \times \left(1+\frac{c_4^2(N_k+\overline{N}_k)}{4P^{(2)}}\right)^2\right]\left(1+\frac{c_4^2c_6(N_1+\overline{N}_1)}{4P^{(2)}}\right)^2,$$

$$L_3 = c_7^2(n_1+\overline{n}_1)^2 \prod_{k=1}^{2}\left[\left(1-\frac{c_4^2(n_k+\overline{n}_k)}{4\varrho^{(1)}}\right)^2 \times \right.$$

$$\left. \times \left(1-\frac{c_4^2(N_k+\overline{N}_k)}{4P^{(1)}}\right)^2\right]\left(1-\frac{c_4^2c_6(n_1+\overline{n}_1)}{4\varrho^{(1)}}\right)^2,$$

$$G_4 = c_7^2(N_1+\overline{N}_1)^2 \prod_{k=1}^{2}\left[\left(1-\frac{c_4^2(n_k+\overline{n}_k)}{4\varrho^{(1)}}\right)^2 \times \right.$$

$$\left. \times \left(1-\frac{c_4^2(N_k+\overline{N}_k)}{4P^{(1)}}\right)^2\right]\left(1-\frac{c_4^2c_6(n_1+\overline{n}_1)}{4P^{(1)}}\right)^2,$$

$$L_4 = c_6^2(n_1+\overline{n})^2 \prod_{k=1}^{2}\left[\left(1+\frac{c_4^2(n_k+\overline{n}_k)}{4\varrho^{(2)}}\right)^2 \times \right.$$

$$\left. \times \left(1+\frac{c_4^2(N_k+\overline{N}_k)}{4P^{(2)}}\right)^2\right]\left(1+\frac{c_4^2c_6(n_1+\overline{n}_1)}{4\varrho^{(2)}}\right)^2.$$

From (11) and (12**) it follows that

$$\text{(20)} \qquad \frac{4}{c_4^2}\sum |n_k|^2 \leqslant \sum (n_k+\overline{n}_k)^2 \leqslant 4\sum |n_k|^2, \qquad \sum \equiv \sum_{k=1}^{2},$$

and since $L(o, c) = (\Sigma|n_k|^2)^{1/2}$, $L(O, C) = (\Sigma N_k^2)^{1/2}$

$$\text{(21)} \qquad \frac{1}{c_4^2e^2} \leqslant \frac{\Sigma|n_k|^2}{c_4^2\Sigma|N_k|^2} \leqslant \frac{\Sigma(n_k+\overline{n}_k)^2}{\Sigma(N_k+\overline{N}_k)^2} \leqslant \frac{c_4^2\Sigma|n_k|^2}{\Sigma|N_k|^2} \leqslant c_4^2e^2.$$

By (12)

$$\text{(22)} \qquad (1+c_7^2)(n_1+\overline{n}_1)^2 \leqslant \Sigma(n_k+\overline{n}_k)^2 \leqslant (1+c_6^2)(n_1+\overline{n}_1)^2$$

$$((1+c_7^2)(N_1+\overline{N}_1)^2 \leqslant \Sigma(N_k+\overline{N}_k)^2 \leqslant (1+c_6^2)(N_1+\overline{N}_1)^2).$$

Thus

$$\frac{1}{c_4^2e^2} \leqslant \frac{(1+c_6^2)(n_1+\overline{n}_1)^2}{(1+c_7^2)(N_1+\overline{N}_1)^2}, \quad \frac{(1+c_7^2)(n_1+\overline{n}_1)^2}{(1+c_6^2)(N_1+\overline{N}_1)^2} \leqslant c_4^2e^2$$

or

$$\text{(23)} \qquad \frac{(1+c_7^2)}{c_4^2e^2(1+c_6^2)} \leqslant \frac{(n_1+\overline{n}_1)^2}{(N_1+\overline{N}_1)^2} \leqslant \frac{(1+c_6^2)c_4^2e^2}{(1+c_7^2)}.$$

For a sufficiently small $(n_1+\bar{n}_1)$ we have

$$(24)\quad \frac{(1+\varepsilon)^{10}c_6^2(1+c_6^2)c_4^2e^4}{(1-\varepsilon)^{10}c_7^2(1+c_7^2)} \geqslant \frac{ds_{\mathfrak{d}}^2(n,\bar{n},du,d\bar{u})}{ds_{\mathfrak{D}}^2(N,\bar{N},dU,d\bar{U})} \geqslant \frac{(1-\varepsilon)^{10}c_7^2}{(1+\varepsilon)^{10}c_6^2e^4}\cdot\frac{(1+c_7^2)}{(1+c_6^2)c_4^2},$$

(see (12) and (12**), p. 80) and hence,

$$(25)\qquad c_3 \geqslant \frac{ds_{\mathfrak{d}}^2(n,\bar{n},du,d\bar{u})}{ds_{\mathfrak{D}}^2(N,\bar{N},dU,d\bar{U})} \geqslant \frac{1}{c_3}, \quad \text{where} \quad c_3 = \frac{e^4c_6^2c_4^2(1+c_6^2)}{c_7^2(1+c_7^2)}+\varepsilon.$$

5. Bounds for $ds_{\mathfrak{d}}/ds_{\mathfrak{D}}$ in the neighborhood of points of the second order. In the present chapter we consider the point $\tilde{o}$ $(\tilde{O})$ through which passes a segment of an analytic surface

$$(1)\qquad \tilde{z}_1 = h(\tilde{z}_2) \qquad (\tilde{Z}_1 = H(\tilde{Z}_2))$$

so that $\tilde{o}$ $(\tilde{O})$ is an interior point of the segment $\tilde{\mathfrak{s}}^2$ and $\tilde{\mathfrak{s}}^2 \in \partial\tilde{\mathfrak{d}}$ $(\tilde{\mathfrak{S}}^2, \tilde{\mathfrak{S}}^2 \in \partial\tilde{\mathfrak{D}})$.

We assume that the transformation

$$(2)\quad t = [z_1 = \tilde{z}_1 - h(\tilde{z}_2), z_2 = \tilde{z}_2] \qquad (T = [Z_1 = \tilde{Z}_1 - H(\tilde{Z}_2), Z_2 = \tilde{Z}_2])$$

is one-to-one and holomorphic in $\tilde{\mathfrak{d}}$ $(\tilde{\mathfrak{D}})$. Let $\mathfrak{d} = t(\tilde{\mathfrak{d}})$ $\big(\mathfrak{D} = T(\tilde{\mathfrak{D}})\big)$. The boundary point $o = t(\tilde{o})$ $(O = T(\tilde{O}))$ of $\mathfrak{d}$ $(\mathfrak{D})$ will lie in the segment $\mathfrak{s}^2$ $(\mathfrak{S}^2)$ of

$$(3)\qquad z_1 = 0 \qquad (Z_1 = 0)$$

and $\mathfrak{s}^2$ $(\mathfrak{S}^2)$ belongs to the boundary of $\mathfrak{d}$ $(\mathfrak{D})$.

The coordinates n_1, n_2 (N_1, N_2) where $n_1 = 0$ $(N_1 = 0)$ coincides with the analytical plane in which the segment $\mathfrak{s}^2$ $(\mathfrak{S}^2)$ is located and where $n_2 = 0$ $(N_2 = 0)$ is the analytic plane perpendicular to $n_1 = 0$ $(N_1 = 0)$, are called coordinates *normal* with respect to the point o (O).

Remark. The plane $n_1 = 0$ $(N_1 = 0)$ coincides with the plane (3).

We make the assumption that

$$(4)\qquad \tilde{\mathfrak{j}} \subset \tilde{\mathfrak{d}} \subset \tilde{\mathfrak{a}} \qquad (\tilde{\mathfrak{J}} \subset \tilde{\mathfrak{D}} \subset \tilde{\mathfrak{A}})$$

where $\tilde{\mathfrak{j}}$ and $\tilde{\mathfrak{a}}$ are domains which one obtains by the PCT t^{-1} from

$$(5)\qquad \mathfrak{j} = [\varrho^{(1)}(n_1+\bar{n}_1) - |n_1|^2 > 0,\ \varrho^{(1)^2} - |n_2|^2 > 0]$$

and

$$(6)\qquad \mathfrak{a} = [\varrho^{(2)}(n_1+\bar{n}_1) + |n_1|^2 > 0,\ \varrho^{(2)^2} - |n_2|^2 < 0]$$

respectively. $\left(\tilde{\mathfrak{J}} = T^{-1}(\mathfrak{J}),\ \tilde{\mathfrak{A}} = T^{-1}(\mathfrak{A}),\right.$ where

$$\mathfrak{J} = [P^{(1)}(N_1+\bar{N}_1)-|N_1|^2 > 0,\ P^{(1)^2}-|N_2|^2 > 0], \tag{7}$$

$$\left.\mathfrak{A} = [P^{(2)}(N_1+\bar{N}_1)+|N_1|^2 > 0,\ P^{(2)^2}-|N_2|^2 < 0]\right). \tag{8}$$

THEOREM 5.1. *Let $\tilde{W}$ be a one-to-one and continuous transformation of the domain $\tilde{\mathfrak{d}}$ onto $\tilde{\mathfrak{D}}$, where $\tilde{\mathfrak{d}}$ and $\tilde{\mathfrak{D}}$ are domains described in Theorem 4.1. Let $\{\tilde{z}^{(K)}\} = \{\tilde{z}_1^{(K)}, \tilde{z}_2^{(K)}\}$ $(\tilde{Z}^{(K)} = \tilde{W}(\tilde{z}^{(K)})$ be a set of points which approach to the point $o = (z_1^0, z_2^0)$ of $\partial\tilde{\mathfrak{d}}$ which is an interior point of the lamina $\tilde{\mathfrak{s}}^2$ $(\tilde{O} = \{\tilde{Z}_1^0, \tilde{Z}_2^0\}$ $\tilde{\mathfrak{S}}^2 \subset \partial\tilde{\mathfrak{D}})$. Further we suppose the points $\{\tilde{z}^{(K)}\}$ $(\{\tilde{Z}^{(K)}\})$, $(\tilde{Z}^{(K)} \to \tilde{Z}^{(0)} = 0)$ lie in the domain*

$$\tilde{\Omega} = t^{-1}\left[|z_1-z_1^0|^2+|z_2-z_2^0|^2 \leqslant \frac{c_2^2}{4}(z_1-z_1^0+\bar{z}_1-\bar{z}_1^0)^2\right] \tag{9}$$

$$\left(\tilde{\Omega} = T^{-1}\left[|Z_1-Z_1^0|^2+|Z_2-Z_2^0|^2 \leqslant \frac{c_2^2}{4}(Z_1-Z_1^0+\bar{Z}_1-\bar{Z}_1^0)^2\right]\right).$$

If the vector $\{d\tilde{u}\}$ $(\{d\tilde{U}\})$ satisfies the hypothesis (3.6) ((3.7)), *see p.* 75, *then there exists a constant $c_3 < \infty$ such that the inequality*

$$0 < \frac{1}{c_3} \leqslant \frac{ds_{\mathfrak{d}}(\tilde{z}, \bar{\tilde{z}}, d\tilde{u}, d\bar{\tilde{u}})}{ds_{\mathfrak{D}}(\tilde{Z}, \bar{\tilde{Z}}, d\tilde{U}, d\bar{\tilde{U}})} \leqslant c_3 \tag{10}$$

holds.

Proof. Since the PCT's t and T preserve the invariant length, we consider the PCT W which transforms $\mathfrak{d}$ into $\mathfrak{D}$. Using the coordinates n_1, n_2 (N_1, N_2) normal with respect to the point o (O), we have by p. 37 of [8]

$$K_{\mathfrak{i}}(n, \bar{n}) = \frac{\varrho^{(1)^2}}{\pi^2\left(n_1+\bar{n}_1-\dfrac{|n_1|^2}{\varrho^{(1)}}\right)^2(\varrho^{(1)^2}-|n_2|^2)^2},$$
$$K_{\mathfrak{a}}(n, \bar{n}) = \frac{\varrho^{(2)^2}}{\pi^2\left(n_1+\bar{n}_1+\dfrac{|n_1|^2}{\varrho^{(2)}}\right)^2(|n_2|^2-\varrho^{(2)^2})^2}; \tag{11}$$

$$ds_{\mathfrak{i}}^2(n, \bar{n}, du, d\bar{u}) = 2\left[\frac{|du_1|^2}{\left(n_1+\bar{n}_1-\dfrac{|n_1|^2}{\varrho^{(1)}}\right)^2}+\frac{\varrho^{(1)^2}|du_2|^2}{(\varrho^{(1)^2}-|n_2|^2)^2}\right],$$
$$ds_{\mathfrak{a}}^2(n, \bar{n}, du, d\bar{u}) = 2\left[\frac{|du_1|^2}{\left(n_1+\bar{n}_1+\dfrac{|n_1|^2}{\varrho^{(2)}}\right)^2}+\frac{\varrho^{(2)^2}|du_2|^2}{(|n_2|^2-\varrho^{(2)^2})^2}\right] \tag{12}$$

and analogous expressions for $K_{\mathfrak{Z}}$, $K_{\mathfrak{A}}$, $ds_{\mathfrak{Z}}$, $ds_{\mathfrak{A}}$ with n_k, $\varrho^{(k)}$, du_k replaced by N_k, $P^{(k)}$, dU_k, $k=1,2$. Thus by (2.1) and (2.2) we obtain

(13) $\tilde{h}_5(n,\bar{n},du,d\bar{u})$

$$\equiv \frac{2\varrho^{(1)^2}\left(1+\dfrac{|n_1|^2}{\varrho^{(2)}h_1}\right)^2(\varrho^{(2)^2}-|n_2|^2)^2\left(|du_1|^2+\dfrac{\varrho^{(1)^2}\left(h_1-\dfrac{|n_1|^2}{\varrho^{(1)}}\right)^2}{(\varrho^{(1)^2}-|n_2|^2)^2}|du_2|^2\right)}{\varrho^{(2)^2}h_1^2\left(1-\dfrac{|n_1|^2}{\varrho^{(1)}h_1}\right)^4(\varrho^{(1)^2}-|n_2|^2)^2}$$

$$\geqslant ds_{\mathfrak{h}}^2(n,\bar{n},du,d\bar{u})$$

$$\geqslant \frac{2\varrho^{(2)^2}\left(1-\dfrac{|n_1|^2}{\varrho^{(1)}h_1}\right)^2(\varrho^{(1)^2}-|n_2|^2)^2\left(|du_1|^2+\dfrac{\varrho^{(2)^2}\left(h_1+\dfrac{|n_1|^2}{\varrho^{(2)}}\right)^2}{(|n_2|^2-\varrho^{(2)^2})^2}|du_2|^2\right)}{\varrho^{(1)^2}h_1^2\left(1+\dfrac{|n_1|^2}{\varrho^{(2)}h_1}\right)^4(|n_2|^2-\varrho^{(2)^2})^2}$$

$$\equiv \tilde{h}_6(n,\bar{n},du,d\bar{u}), \qquad h_1\equiv n_1+\bar{n}_1.$$

Using (13) and the analogous formulas for $\tilde{H}_k(N,\bar{N},dU,d\bar{U})$, $k=5,6$, yields

(14)
$$\frac{\varrho^{(1)^2}P^{(1)^2}H_1^2\left(1+\dfrac{|n_1|^2}{\varrho^{(2)}h_1}\right)^2(\varrho^{(2)^2}-|n_2|^2)^2\left(1+\dfrac{|N_1|^2}{P^{(2)}H_1}\right)^4(|N_2|^2-P^{(2)^2})^2}{\varrho^{(2)^2}P^{(2)^2}h_1^2\left(1-\dfrac{|n_1|^2}{\varrho^{(1)}h_1}\right)^4(\varrho^{(1)^2}-|n_2|^2)^2\left(1-\dfrac{|N_1|^2}{P^{(1)}H_1}\right)^2(|N_2|^2-P^{(1)^2})^2}\times$$

$$\times\left(\frac{|du_1|^2+\dfrac{\varrho^{(1)^2}h_1^2\left(1-\dfrac{|n_1|^2}{\varrho^{(1)}h_1}\right)}{(\varrho^{(1)^2}-|n_2|^2)^2}|du_2|^2}{|dU_1|^2+\dfrac{P^{(2)^2}H_1^2\left(1+\dfrac{|N_1|^2}{P^{(2)}H_1}\right)^2}{(P^{(2)^2}-|N_2|^2)^2}|dU_2|^2}\right)\geqslant\frac{ds_{\mathfrak{h}}^2(n,\bar{n},du,d\bar{u})}{ds_{\mathfrak{D}}^2(N,\bar{N},dU,d\bar{U})}$$

$$\geqslant\frac{\varrho^{(2)^2}P^{(2)^2}H_1^2\left(1-\dfrac{|n_1|^2}{\varrho^{(1)}h_1}\right)^2(\varrho^{(1)^2}-|n_2|^2)^2\left(1-\dfrac{|N_1|^2}{P^{(1)}H_1}\right)^4(P^{(1)^2}-|N_2|^2)^2}{\varrho^{(1)^2}P^{(1)^2}h_1^2\left(1+\dfrac{|n_1|^2}{\varrho^{(2)}h_1}\right)^4(|n_2|^2-\varrho^{(2)^2})^2\left(1+\dfrac{|N_1|^2}{P^{(2)}H_1}\right)^2(P^{(2)^2}-|N_2|^2)^2}\times$$

$$\times\left(\frac{|du_1|^2+\dfrac{\varrho^{(2)^2}h_1^2\left(1+\dfrac{|n_1|^2}{\varrho^{(2)}h_1}\right)^2|du_2|^2}{(|n_2|^2-\varrho^{(2)^2})^2}}{|dU_1|^2+\dfrac{P^{(1)^2}H_1^2\left(1-\dfrac{|N_1|^2}{P^{(1)}H_1}\right)^2}{(P^{(1)^2}-|N_2|^2)^2}|dU_2|^2}\right),\qquad \text{where } H_1=N_1+\bar{N}_1.$$

Since we assume that we approach to o (O) in the cone Ω, it holds by (3.13), p. 76,

$$\frac{1}{4}(n_1+\bar{n}_1)^2 \leqslant |n_1|^2+|n_2|^2 \leqslant \frac{1}{4}c_2^2(n_1+\bar{n}_1)^2 \tag{15}$$

and an analogous inequality for $(N_1+\bar{N}_1)^2$. Thus for sufficiently small $n_1, n_2 (N_1, N_2)$, by (15) and (3.14),

$$\frac{\varrho^{(1)^2}P^{(1)^2}c_2^2(|N_1|^2+|N_2|^2)(1+\varepsilon)^2\varrho^{(2)^4}(1+\varepsilon)^4P^{(2)^4}}{\varrho^{(2)^2}P^{(2)^2}(|n_1|^2+|n_2|^2)(1-\varepsilon)^4(\varrho^{(1)^2}-\varepsilon)^2(1-\varepsilon)^2(P^{(1)^2}-\varepsilon)^2}\times \tag{16}$$

$$\times\left(\frac{|du_1|^2+\dfrac{(n_1+\bar{n}_1)^2\varrho^{(1)^2}\left(\dfrac{1}{c_1^2}-1\right)|du_1|^2}{(\varrho^{(1)^2}-|n_2|^2)^2}}{|dU_1|^2+\dfrac{(N_1+\bar{N}_1)^2}{P^{(2)^2}}\left(\dfrac{1}{c_5^2}-1\right)|dU_1|^2}\right) \geqslant \frac{ds_{\mathfrak{d}}^2(n,\bar{n},du,d\bar{u})}{ds_{\mathfrak{D}}^2(N,\bar{N},dU,d\bar{U})}$$

$$\geqslant \frac{\varrho^{(1)^2}P^{(2)^2}(|N_1|^2+|N_2|^2)(1-\varepsilon)^2(\varrho^{(1)^2}-\varepsilon)^2(1-\varepsilon)^4(P^{(1)^2}-\varepsilon)^2}{\varrho^{(1)^2}P^{(1)^2}c_2^2(|n_1|^2+|n_2|^2)(1+\varepsilon)^4\varrho^{(2)^4}(1+\varepsilon)^2P^{(2)^4}}\times$$

$$\times\left(\frac{|du_1|^2+\dfrac{(n_1+\bar{n}_1)^2\varrho^{(2)^2}\left(\dfrac{1}{c_5^2}-1\right)|du_1|^2}{\varrho^{(2)^4}}}{|dU_1|^2+\dfrac{(N_1+\bar{N}_1)^2P^{(1)^2}\left(\dfrac{1}{c_1^2}-1\right)|dU_1|^2}{(P^{(1)^2}-\varepsilon)^2}}\right), \qquad \varepsilon>0.$$

From (3.6), (3.7) and (3.8), p. 75, it follows that

$$\frac{c_1^2}{c_5^2e^2} \leqslant \frac{c_1^2(|du_1|^2+|du_2|^2)}{c_5^2(|dU_1|^2+|dU_2|^2)} \leqslant \left|\frac{du_1}{dU_1}\right|^2 \leqslant \frac{c_5^2(|du_1|^2+|du_2|^2)}{c_1^2(|dU_2|^2+|dU_2|^2)} \leqslant \frac{c_5^2e^2}{c_1^2}. \tag{17}$$

Using the results of §§ 3 and 4 and carrying out some additional considerations (see [10a]), we obtain bounds (1.3) in domains bounded by two segments of analytic hypersurfaces.

5. Description of a special class of QPCT's W. In order to obtain the desired bounds for the distortion of non-Euclidian measures it is sufficient to define a diffeomorphism H of a domain $\mathfrak{d}$ onto $\mathfrak{D}$. Since the metric ds^2 is invariant under PCT's it is convenient to choose among pseudoconformally equivalent domains, a domain of a simple structure, and to determine bounds for the distortion of euclidian measures under the diffeomorphism H. For instance if $\mathfrak{D}$ is a domain which is pseudoconformally equivalent to a circular domain $\mathfrak{C}$, we shall consider a mapping of the unit hypersphere $\mathfrak{c}$ onto $\mathfrak{C}$.

In [9] a QPCT of $\mathfrak{c}$ onto a Reinhardt circular domain has been discussed. (It should be noted that the present QPCT is different from that considered in [9] for the Reinhardt circular domains.)

For every circular domain $\mathfrak{C}$ we can construct its "representant" $\mathfrak{r}(\mathfrak{C})$. This is the totality of points (X_1, Y_1, X_2, Y_2) of $\mathfrak{C}$ for which $Y_2 = 0$ and $X_2 > 0$. On the other hand, with every point $Z_1 = R_1 e^{i\Phi_1}$, X_2 of $\mathfrak{r}(\mathfrak{C}) + \mathfrak{C} \cap (Z_2 = 0)$ we associate the "orbit":

$$\mathfrak{T}(Z_1, X_2) = [Z_1 = X_1 e^{i(\Phi_1+\Phi)},\ Z_2 = X_2 e^{i\Phi},\ 0 \leqslant \Phi \leqslant 2\pi]. \tag{1}$$

Then

$$\mathfrak{C} = \bigcup \mathfrak{T}(Z_1, X_2) \quad \text{where} \quad (Z_1, X_2) \in \mathfrak{r}(\mathfrak{C}) + \mathfrak{C} \cap (Z_2 = 0). \tag{2}$$

In the following we assume

a) there exists a boundary point $P(Z_1^{(p)}, X_2^{(p)})$ of $\bar{\mathfrak{r}}(\mathfrak{C})$ such that

$$X_2^{(p)} > X_2, \qquad (Z_1, X_2) \in \bar{\mathfrak{r}}(\mathfrak{C}) - P; \tag{3}$$

b) the segment of the straight line connecting the origin O with P, i.e.

$$Z_1 = r Z_1^{(p)}, \qquad X_2 = r X_2^{(p)}, \qquad 0 < r < 1, \tag{4}$$

lies in $\mathfrak{r}(\mathfrak{C})$;

c) every intersection

$$\mathfrak{r}(\mathfrak{C}) \cap (X_2 = \text{const}), \qquad X_2 < X_2^{(p)}, \tag{5}$$

is a simply connected domain.

We define the QPCT W of the unit hypersphere $\mathfrak{c}$ onto the circular domain $\mathfrak{C}$ by describing at first the mapping of $\mathfrak{r}(\mathfrak{c}) + \mathfrak{c} \cap (z_2 = 0)$ onto $\mathfrak{r}(\mathfrak{C}) + \mathfrak{C} \cap (Z_2 = 0)$. We set

$$X_2 = s x_2, \qquad s = X_2^{(p)}, \tag{6}$$

and we assume that the circle $[|z_1| \leqslant (1 - x_2^{*2})^{1/2}, x_2 = x_2^*]$, is mapped conformally onto

$$\mathfrak{r}(\mathfrak{C}) \cap (X_2 = s x_2^*) \tag{7}$$

so that the point $[z_1 = 0, x_2 = x_2^*]$ goes into $(Z_1 = s z_1, X_2 = s x_2^*)$ of $\mathfrak{r}(\mathfrak{C})$ and the direction of the positive x_1-axis goes into the direction of the positive X_1-axis.

The mapping W of $\mathfrak{c}$ onto $\mathfrak{C}$ is defined by assuming that (z_1^*, z_2^*) goes into (Z_1^*, Z_2^*) where

$$z_1^* = z_1 e^{i\varphi}, \qquad z_2^* = x_2 e^{i\varphi}, \tag{8}$$

$$Z_1^* = Z_1 e^{i\Phi}, \qquad Z_2 = X_2 e^{i\Phi}, \tag{9}$$

$(z_1, x_2) \in \mathfrak{r}(\mathfrak{c}) + \mathfrak{c} \cap (z_2 = 0), (Z_1, X_2) \in \mathfrak{r}(\mathfrak{C}) + \mathfrak{C} \cap (Z_2 = 0)$, and

$$\varphi = \Phi. \tag{10}$$

Let $du = (du_1, du_2)$ be a vector which is transformed by the QPCT W into $dU = (dU_1, dU_2)$. We shall determine bounds for the distortion of euclidean length of du, i.e. bounds for

$$\left(\frac{|dU_1|^2 + |dU_2|^2}{|du_1|^2 + |du_2|^2}\right)^{1/2}. \tag{11}$$

Since by the transformation $z_k^* = z_k e^{i\varphi} (Z_k^* = Z_k e^{i\Phi})$, $k = 1, 2$, the domain $\mathfrak{c}(\mathfrak{C})$ is transformed onto itself and since by a rotation the Euclidean length is unchanged, we can assume that the one endpoint (z_1, z_2), $((Z_1, Z_2))$ of the vector $du(dU)$ lies in $[x_2 \geqslant 0, y_2 = 0]$, $(X_2 \geqslant 0, Y_2 = 0)$.

Since we assume that W transforms $\mathfrak{c} \cap (x_2 = \text{const.}, y_2 = 0) \equiv \mathfrak{c}(x_2)$ conformally onto $\mathfrak{C} \cap (X_2 = sx_2, Y_2 = 0) = \mathfrak{C}(sx_2)$, it holds

$$|dU_1|^2 = |du_1|^2 \frac{K_{\mathfrak{c}(x_2)}(z_1, \bar{z}_1)}{K_{\mathfrak{C}(sx_2)}(Z_1, \bar{Z}_1)} \tag{12}$$

$K_{\mathfrak{c}(x_2)}$ is the kernelfunction of $\mathfrak{c}(x_2)$. If we write

$$du_2 = dt_2 + idv_2, \qquad dU_2 = dT_2 + idV_2, \tag{13}$$

then

$$\frac{dT_2}{dt_2} = s, \tag{14}$$

$$|du_2| \cos\varphi = dt_2, \qquad |du_2| \sin\varphi = dv_2 \tag{15}$$

and by (10)

$$|dU_2| \cos\varphi = dT_2, \qquad |dU_2| \sin\varphi = dV_2. \tag{16}$$

Consequently

$$dV_2 = dT_2 \frac{dv_2}{dt_2} = s dv_2, \tag{17}$$

$$|dU_1|^2 + |dU_2|^2 = |du_1|^2 \frac{K_{\mathfrak{c}(x_2)}(z_1, \bar{z}_1)}{K_{\mathfrak{C}(sx_2)}(Z_1, Z_2)} + s^2 |du_2|^2. \tag{18}$$

Thus:

Let the QPCTT maps the hypersphere $\mathfrak{c}$ onto the circular domain $\mathfrak{C}$ and the vector (du_1, du_2) at the point (z_1, z_2) into the vector (dU_1, dU_2) at the point (Z_1, Z_2) as described above. Then for the distortion of the Euclidean length holds

$$\left(\frac{|dU_1|^2 + |dU_2|^2}{|du_1|^2 + |du_2|^2}\right)^{\frac{1}{2}} \leqslant \left(\max\left[\frac{K_{\mathfrak{c}(x_2)}(z_1, \bar{z})}{K_{\mathfrak{C}(sx_2)}(Z_1, \bar{Z}_1)}, s^2\right]^{1/2}\right). \tag{19}$$

Here $K_{\mathfrak{c}(x_2^*)}(z_1, \bar{z}_1) \big(K_{\mathfrak{C}(sx_2^*)}(Z_1, \bar{Z}_1)\big)$ is the kernel function of the intersection of $\mathfrak{c}$ with the plane $(x_2 = x_2^*, y_2 = 0)$ (of $\mathfrak{C}$ with $(X_2 = sx_2^*, y_2 = 0)$).

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Reçu par la Rédaction le 8. 4. 1966