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ON THE HOLOMORPHISM OF THE INTEGRAL WITH RESPECT TO A COMPLEX PARAMETER

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M. Warmus has raised the following question: Let the function

$$F(x,z) = \sum_{n=0}^{\infty} F_n(x) z^n \tag{1}$$

be holomorphic in the circle $|z| < 1$ for any x from the interval $\langle 0, 1\rangle$ and integrable in the interval $\langle 0, 1\rangle$ for any z from the circle $|z| < 1$. The question is whether under the above assumptions the functions $F_n(x)$ are integrable [1].

In the sequel we shall give an example of a function satisfying the above conditions and such that the function $F_1(x)$ is non-integrable.

Put for $n = 1, 2, \ldots$

$$A_n = \{z\colon 4^{-n} \leqslant |z| \leqslant 2^n;\ |\operatorname{Arg} z - 2^{-n}| \leqslant 4^{-n}\} \tag{2}$$

and

$$B_n = \{z\colon |z| \leqslant 2^{-1-2n}\}. \tag{3}$$

Let $P_n(z)$ be a polynomial satisfying the conditions

$$|P_n(z) - 1| \leqslant 2^{-1-2n} \quad \text{for} \quad z \in B_n \tag{4}$$

and

$$|P_n(z)| \leqslant 2^{-1-2n} \quad \text{for} \quad z \in A_n. \tag{5}$$

The existence of a polynomial P_n with the required properties follows from Runge Theorem on approximation of holomorphic functions by polynomials.

Put further

$$F(x,z) = z 2^n \big(P_n(z) - P_n(0) + 1\big) \quad \text{for} \quad 2^{-n} < x \leqslant 2^{-1+n};\ n = 1, 2, \ldots \tag{6}$$

(1) M. Warmus, *Problem 142*, Colloquium Mathematicum 3 (1955), p. 173.

It immediately follows from the definition that function (6) is holomorphic. We will show that *for a fixed z this function is integrable with respect to x.* First we observe that

(7) $$\int_0^1 F(x, 0)\,dx = 0.$$

Thus let $z \neq 0$. There exists a positive integer N such that $z \in A_n$ for $n \geqslant N$. According to (2), (5) and (6) we thus obtain

(8) $$\int_0^{2^{-N}} |F(x, z)|\,dx = \sum_{n=N}^{\infty} \int_{2^{-n-1}}^{2^{-n}} |F(x, z)|\,dx \leqslant \sum_{n=N}^{\infty} 2^{-n} < \infty.$$

In view of (6) we also have

(9) $$\int_{2^{-N}}^{1} |F(x, z)|\,dx \leqslant 2^N |z| \operatorname*{Max}_{1\leqslant n\leqslant N} \big(2^{-3}+|P_n(z)|\big) < +\infty.$$

From (8) and (9) it follows that

(10) $$\int_0^1 |F(x, z)|\,dx < \infty$$

for every complex z. It is seen from formula (6) that the coefficient $F_1(x)$ in development (1) of function (6) is given by the formulae

(11) $$F_1(x) = F'_z(x, 0) = 2^n \quad \text{for} \quad 2^{-n} < x \leqslant 2^{1-n}.$$

Thus

(12) $$\int_0^1 |F_1(x)|\,dx = \infty.$$

It follows from formula (12) that the answer to Mr. Warmus question is negative.

The question dealt with above gives rise to the question of whether the integral of function (1) is holomorphic. The example given above implies a negative answer also to this question.

To this end put

(13) $$g(z) = \int_0^1 F(x, z)\,dx$$

and consider $g(z)$ in a neighbourhood of 0. By (7) we have $g(0) = 0$. Now by (2)-(6) we get

(14) $$|g(z_n)| \geqslant (n-2)z_n$$

for $z_n = 4^{-n}$. Thus

$$|g(z_n)-g(0)| \geqslant (n-2)z_n,$$

which means that the function g is non-differentiable at $z = 0$. One may obtain a positive answer to both questions dealt with above if one assumes in addition that integrals (10) are bounded.

Namely, the following lemma is true:

LEMMA. *Let the function*

$$F(x,z) = F_0(x)+F_1(x)(z-z_0)+F_2(x)(z-z_0)^2+\dots \tag{15}$$

be holomorphic in the circle $|z-z_0| < R$ *for every* $x \in \langle 0,1\rangle$ *and let*

$$\int_0^1 |F(x,z)|\,dx \leqslant M(r) \tag{16}$$

for every z *such that* $0 < |z-z_0| = r < R$, *where* $M(r)$ *is a constant depending solely on* r *and the function* F.

Then the function

$$g(z) = \int_0^1 F(x,z)\,dx \tag{17}$$

is holomorphic in the circle $|z-z_0| < r$ *and one obtains its development into a power series by integrating the series* (15) *term by term in the interval* $0 \leqslant x \leqslant 1$, *the functions* F_n *being integrable.*

Proof. Without loss of generality we may assume that $z_0 = 0$. By (16) we obtain

$$\int_0^1 \int_0^{2\pi} \frac{|F(x, re^{it})|}{|re^{it}-z|^{n+1}}\, r\,dx\,dt \leqslant \frac{2\pi M(r)}{|z-r|^{n+1}} \tag{18}$$

for $|z| < r$ and $n = 0, 1, 2, \dots$ Thus according to the Fubini theorem we have

$$g(z) = \int_0^1 F(x,z)\,dx = \int_0^1 \left(\frac{1}{2\pi i}\int_{C_r} \frac{F(x,\mathfrak{z})}{\mathfrak{z}-z}\,d\mathfrak{z}\right)dx \tag{19}$$

$$= \frac{1}{2\pi i}\int_{C_r}\left(\int_0^1 \frac{F(x,\mathfrak{z})}{\mathfrak{z}-z}\,dx\right)d\mathfrak{z} = \frac{1}{2\pi i}\int_{C_r}\frac{g(\mathfrak{z})}{\mathfrak{z}-z}\,d\mathfrak{z},$$

where C_r is the circumference $|\mathfrak{z}| = r$. Thus, the function g has been expressed by the Cauchy integral formula and consequently it is holomorphic in the circle $|z| < r$.

Further, from inequality (18) we obtain analogously as (19) that

$$\int_0^1 |F_n(x)|\,dx \leqslant \frac{M(r)}{r^n} \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{20}$$

This means that the functions $F_n(x)$ are integrable and that the series

$$\sum_{n=0}^{\infty} \int_0^1 |F_n(x) z^n|\,dx \tag{21}$$

converges for $|z| < r$.

This statement together with (15) and (19) implies

$$g(z) = \sum_{n=0}^{\infty} z^n \int_0^1 F_n(x)\,dx, \tag{22}$$

which ends the proof of the lemma.

If we assume only that integrals (16) converge, without assuming that they are bounded, we obtain the following result:

THEOREM. *Let $F(x, z)$ be a function holomorphic in a domain G for every $x \in \langle 0, 1\rangle$ and integrable in the interval $0 \leqslant x \leqslant 1$ for every $z \in G$. Then there exists an open set $G^* \subset G$, dense in G and such that the function*

$$g(z) = \int_0^1 F(x, z)\,dx$$

is holomorphic in G^*. Moreover, for every $z_0 \in G^*$ there exists an $r > 0$ such that

$$\int_0^1 F(x, z)\,dx = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} \int_0^1 F_{z^n}^{(n)}(x, z_0)\,dx \tag{23}$$

for $|z-z_0| < r$.

Proof. Consider an arbitrary closed circle $K \subset G$ and put

$$E_n = \left\{z \colon z \in K \text{ and } \int_0^1 |F(x, z)|\,dx \leqslant n\right\} \tag{24}$$

for $n = 1, 2, \ldots$ The sets E_n are closed and $\bigcup E_n = K$. Thus by Baire theorem on a complete space at least one of the sets E_n includes a circle K^*. Thus by the above lemma the function $g(z)$ is holomorphic in K^*, and for every $z_0 \in K^*$ there exist a positive number r such that relation (23) holds. In addition we easily prove that the set

$$G^* = \bigcup_{K \subset G} K^*, \tag{25}$$

where K varies in the family of closed circles included in G and K^* has a meaning described above, has the properties asserted in the statement of the theorem. (K^* has not been defined uniquely. In the sum of sets (25) one may take for K^* any one of the sets corresponding to K by the above argument.) Thus the theorem has been proved.

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