COLLOQUIUM MATHEMATICUM

VOL. XX 1969 FASC. 2

FUNCTIONS REPRESENTED BY INTEGRATED RADEMACHER SERIES

BY

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Functions represented by lacunary trigonometric series provide interesting examples of certain types of functions, such as the Weierstrass function $f(t) = \sum_{n=1}^{\infty} 2^{-n}\cos 2^n t$ which is continuous but yet differentiable nowhere ([18], p. 47). Several authors have considered functions represented by series of "saw-tooth" functions which also provide interesting examples of continuous nowhere-differentiable functions ([3], p. 115; [7], p. 38; [17], p. 353) and continuous functions having a symmetric derivative everywhere but non-differentiable on a dense set ([15], p. 93).

The present author has noticed that many of the series of "saw-tooth" functions previously considered are simply integrated Rademacher series. Also there is a close parallel between trigonometric series of the form

$$\sum_{m=1}^{\infty} a_m \int_0^t \sin(b_m x)\,dx = \sum_{m=1}^{\infty} a_m b_m^{-1}[1-\cos b_m t] \tag{1}$$

and integrated Rademacher type series of the form

$$\sum_{m=1}^{\infty} a_m \int_0^t r_1(b_m x)\,dx \equiv \sum_{m=1}^{\infty} a_m s_m(t), \tag{2}$$

where $\{a_m\}$ is a sequence of real numbers, $\{b_m\}$ is an increasing sequence of positive integers such that b_m/b_{m-1} is even, and $r_1(t)$ is defined as follows:

$$r_1(t) = \begin{cases} 1, & 0 \leqslant t < 1/2, \\ -1, & 1/2 \leqslant t < 1, \\ r_1(t+k), & \text{for any integer } k. \end{cases} \tag{3}$$

In the simplest case, $b_m = 2^{m-1}$, $s_m(t)$ is the primitive of $r_m(t) = r_1(2^{m-1}t)$, the m-th Rademacher function. The main reason for this parallel appears to be the fact that $r_1(t) = \operatorname{sign}\sin \pi t$ whenever the latter is non-zero.

In this paper we study series of the form (2) and find analogues of known theorems on lacunary trigonometric series of the form (1). In general, the proofs of our theorems are simpler than either the corresponding ones for trigonometric series or for the series of "saw-tooth" functions previously considered. For instance, Theorem 3 generalizes all the well-known examples of continuous nowhere-differentiable functions which are represented by series of "saw-tooth" functions, and yet its proof is no more difficult than those previously given for the special cases. Also, by utilizing integrated Rademacher series we give a simple example of an absolutely continuous function possessing a proper minimum on a dense set (Corollary 4).

Now let $F(t)$ denote the sum of series (2) whenever it exists. Since $|s_m(t)| < b_m^{-1}$, we have

LEMMA 1. *If* $\sum |a_m b_m^{-1}| < \infty$, *then* F *is continuous.*

Also, since $r_1(b_m t)$ is constant over $[p/b_k, (p+1)/b_k)$ for $m < k$ and orthogonal over that interval for $m \geqslant k$, we have

LEMMA 2. *If* $p/b_k \leqslant t < (p+1)/b_k$, *where* p *and* k *are integers,* $k > 1$, *then*

$$F[(p+1)/b_k] - F[p/b_k] = b_k^{-1} \sum_{m=1}^{k-1} a_m r_1(b_m t). \tag{4}$$

In order to study the modulus of continuity of F (defined in [18], p. 42) we will need

LEMMA 3. *Let* $\Phi_1(t)$ *be orthogonal and bounded on* $[0,1)$, $\Phi_1(t+k) = \Phi_1(t)$, *and* $\Phi_m(t) = \Phi_1(mt)$, *where* k *and* $m = 1, 2, \ldots$ *If* $0 < h \leqslant 1$ *and*

$$G(t) = \sum_{m=1}^{\infty} a_m \int_0^t \Phi_m(x)\,dx$$

exists for all t, *then*

$$|G(t+h) - G(t)| = O\Big[h \sum_{m=1}^{2^n} |a_m| + \sum_{m=2^n+1}^{\infty} |a_m| m^{-1}\Big] \tag{5}$$

uniformly in t, *where* n *is chosen such that* $2^{-n-1} < h \leqslant 2^{-n}$.

Proof.

$$G(t+h) - G(h) = \sum_{m=1}^{\infty} a_m \int_t^{t+h} \Phi_m(x)\,dx = \sum_{m=1}^{2^n} + \sum_{m=2^n+1}^{\infty} = P + Q,$$

$$|P| = O\Big[h \sum_{m=1}^{2^n} |a_m|\Big], \qquad |Q| = O\Big[\sum_{m=2^n+1}^{\infty} |a_m| m^{-1}\Big].$$

THEOREM 1. *Let $G(t)$ be as defined in Lemma 3. If $0 < a \leqslant 1$ and*

$$\sum_{m=1}^{n} |a_m| = O[n^{(1-a)}],$$

then $G \in \mathrm{Lip}\, a$.

Proof. By Lemma 3 we have

$$|G(t+h)-G(t)| = O\Big[h2^{n(1-a)}+\sum_{k=n}^{\infty}2^{-ak}\Big] = O[h^a].$$

COROLLARY 1 (G. G. Lorentz [2], p. 217). *If $0 < a < 1$, and*

$$\sum_{m=n}^{\infty}(|a_m|+|b_m|) = O(n^{-a}), \tag{6}$$

then $H(t) = \sum_{m=1}^{\infty}(a_m \cos mt + b_m \sin mt) \in \mathrm{Lip}\, a$.

Proof. Condition (6) is equivalent to

$$\sum_{m=1}^{n}(|a_m|+|b_m|)m = O[n^{(1-a)}], \quad 0 < a < 1.$$

We now return to series of the form (2) and prove

THEOREM 2. *If $0 < a \leqslant 1$, then $F \in \mathrm{Lip}\, a$ if and only if*

$$\sum_{m=1}^{k-1} |a_m| = O[b_k^{(1-a)}].$$

Proof. If $F \in \mathrm{Lip}\, a$ and t is chosen such that $r_1(b_m t) = \mathrm{sign}\, a_m$ for $1 \leqslant m < k$, then by Lemma 2

$$b_k^{-1}\sum_{m=1}^{k-1} |a_m| = O[b_k^{-a}].$$

This together with Theorem 1 implies our desired result.

COROLLARY 2. *If $0 < a < 1$, then $F \in \mathrm{Lip}\, a$ if and only if $|a_m| = O[b_m^{(1-a)}]$.*

COROLLARY 3. *$F \in \mathrm{Lip}\, 1$ if and only if $\sum |a_m| < \infty$.*

THEOREM 3. *If $\sum |a_m b_m^{-1}| < \infty$ and $a_m \neq o(1)$, then F is continuous but has no finite derivative anywhere.*

Proof. If $p/b_k \leqslant t < (p+1)/b_k$, then

$$\{F[(p+1)/b_k]-F[p/b_k]\}b_k = \sum_{m=1}^{k-1} a_m r_1(b_m t) \tag{7}$$

cannot converge for a fixed t as $k \to \infty$. But it follows easily (cf. [3], p. 115) that if $a_n \leqslant t < \beta_n$ for $n = 1, 2, \ldots, \beta_n - a_n \to 0$ as $n \to \infty$, and

$f'(t)$ exists and is finite, then

$$[f(\beta_n)-f(\alpha_n)]/[\beta_n-\alpha_n] \to f'(t) \quad \text{as } n \to \infty;$$

and hence if $F'(t)$ exists and is finite we must have that (7) converges.

Remark 1. For $b_n = 2^{m-1}$, Theorem 3 was previously proved for several special cases such as $a_m \equiv 1$ ([3], p. 115) and $a_m = 1$ for $m = 2n+1$, $n = 0, 1, 2, \ldots$ and 0 otherwise ([7], p. 38). Series of this type were first considered by G. Faber ([16], pp. 546-554) and hence may be called *Faber series*. In fact, the case $a_m = 10^{-n}2^{n!}$, for $m = 2^{n!}$, $n = 1, 2, \ldots$, and 0 otherwise was given by Faber ([5], p. 538) in 1907 before the Rademacher functions were invented! The case $b_m = 10^m$ and $a_m \equiv 1$ was given by B. van der Waerden in 1930 ([17], p. 353).

By a slight modification of the proof of Theorem 3 we now prove the stronger

THEOREM 4. *If $\sum |a_m b_m^{-1}| < \infty$ and $a_m \neq o(1)$, then F is continuous but has neither a finite right- or left-hand derivative at any point.*

Proof. If $p/b_k \leqslant t < (p+1)/b_k$, then

$$\{F[(p+2)/b_k]-F[(p+1)/b_k]\}b_k = \sum_{m=1}^{k-1} a_m r_1[b_m(p+1)/b_k] \tag{8}$$

cannot converge for a fixed t as $k \to \infty$. But it follows easily (cf. [3], p. 115) that if $t < \alpha_n < \beta_n$ for $n = 1, 2, \ldots, \beta_n - t \to 0$, and the sequence $\{(\beta_n - t)/(\beta_n - \alpha_n)\}$ is bounded, then

$$[f(\beta_n)-f(\alpha_n)]/[\beta_n-\alpha_n] \to f(t+) \quad \text{as } n \to \infty$$

whenever $f'(t+)$ exists and is finite; and thus (8) approaches $F'(t+)$ whenever the right-hand derivative exists and is finite. A similar argument shows that the left-hand derivative exists nowhere.

Remark 2. McShane and Botts proved Theorem 4 ([14], p. 116) for the case $b_m = 12^{m-1}$ and $a_m = 6^{m-1}$, $m = 1, 2, \ldots$

Remark 3. We may not omit the condition of finiteness in Theorem 4. To verify this we may consider the function $\mathscr{F}_1(t) = \sum_{m=1}^{\infty} \int_0^t r_m(x)\,dx$. Choose now $0 < h \leqslant 1$ and $2^{-k-1} < h \leqslant 2^{-k}$. Then

$$[\mathscr{F}_1(h)-\mathscr{F}_1(0)]/h \geqslant \sum_{m=1}^{k-1} h^{-1} \int_0^h r_m(x)\,dx = \sum_{m=1}^{k-1} 1.$$

Thus $\mathscr{F}_1'(0+) = +\infty$; similarly $\mathscr{F}_1'(0-) = -\infty$. From these results it follows that $\mathscr{F}_1(t)$ has at all dyadic rationals (i.e., numbers of the foım $p/2^k$) a right-hand derivative equal to $+\infty$ and a left-hand derivative equal to $-\infty$.

THEOREM 5. *If* $a_m \neq o(1), a_m = O(1)$, *then* F *has neither a finite right- nor left-hand derivative at any point and*

$$\omega(F, h) = O[h \log h],$$

where $\omega(F, h)$ *denotes the modulus of continuity of* F.

Proof. By Lemma 3, if $2^{-n-1} < h \leqslant 2^{-n}$

$$|F(t+h)-F(t)| = O[hn+2^{-n}] = O[h\log h].$$

Remark 4. By Theorem 2, 3, 4 and 5 and Remark 3 we see that

$$F_a(t) = \sum_{m=1}^{\infty} b^{(1-a)m} \int_0^t r_1(b^m x)\,dx,$$

where b is an even positive integer and $0 < a \leqslant 1$, is quite similar to the Weierstrass function ([18], pp. 44-48; [9], p. 170)

$$\sum_{m=1}^{\infty} b^{(1-a)m} \int_0^t \sin b^m x\,dx = 1 - \sum_{m=1}^{\infty} b^{-am} \cos b^m t.$$

However, $F_1(t) \notin \Lambda_*$ (cf. [18], p. 47) since

$$F_1(b^{-n})+F_1(-b^{-n})-2F_1(0) = \sum_{m=1}^{\infty}\Big[\int_0^{b^{-n}} r_1(b^m x)\,dx - \int_{-b^{-n}}^{0} r_1(b^m x)\,dx\Big]$$

$$= \sum_{m=1}^{n-1} [b^{-n}+b^{-n}] = 2(n-1)b^{-n} \neq O(b^{-n}).$$

We now proceed to

THEOREM 6. *If* $\sum |a_m b_m^{-1}| < \infty$ *and*

$$\limsup\Big\{|a_k|3^{-1} - \sum_{m=1}^{k-1} |a_m|\Big\} = +\infty, \tag{9}$$

then $F(t)$ *is continuous and has upper and lower derivates of* $+\infty$ *and* $-\infty$, *respectively, for every* t.

Proof (cf. [14], pp. 113-115). Fix t and choose $t_2 = p/b_{k+1} \leqslant t < (p+1)/b_{k+1} = t_3$, $t_1 = (p-1)/b_{k+1}$, and $t_4 = (p+2)/b_{k+1}$, where p is an integer. Then

$$\frac{F(t_3)-F(t_2)}{t_3-t_2} = \sum_{m=1}^{k} a_m r_1(b_m t) = a_k r_k(b_k t) + \sum_{m=1}^{k-1} a_m r_1(b_m t), \tag{10}$$

$$\frac{F(t_4)-F(t_1)}{t_4-t_1} = \Big[a_k \int_{t_3}^{t_4} r_1(b_k x)\,dx + \sum_{m=1}^{k-1} a_m \int_{x_1}^{x_4} r_1(b_m x)\,dx\Big] b_k 3^{-1} \tag{11}$$

$$= -a_k r_k(b_k t)3^{-1} + \Big[\sum_{m=1}^{k-1}\Big] b_k 3^{-1}.$$

Suppose now that $a_k r_k(b_k t) = |a_k|$. Then by (10) and (11) we have

$$\frac{F(t_3)-F(t_2)}{t_3-t_2} \geqslant |a_k| - \sum_{m=1}^{k-1} |a_m|, \tag{12}$$

$$\frac{F(t_4)-F(t_1)}{t_4-t_1} \leqslant -|a_k|3^{-1} + \sum_{m=1}^{k-1} |a_m|. \tag{13}$$

Now since

$$\frac{F(t_4)-F(t_1)}{t_4-t_1} = \frac{t-t_1}{t_4-t_1}\cdot\frac{F(t)-F(t_1)}{t-t_1} + \frac{t_4-t}{t_4-t_1}\cdot\frac{F(t_4)-F(t)}{t_4-t},$$

we must have either

$$\frac{F(t)-F(t_1)}{t-t_1} \quad \text{or} \quad \frac{F(t_4)-F(t)}{t_4-t}$$

less than or equal to the right-hand side of (13). Similarly either

$$\frac{F(t)-F(t_2)}{t-t_2} \quad \text{or} \quad \frac{F(t_3)-F(t)}{t_3-t}$$

is greater than or equal to the right-hand side of (12).

An analogous argument holds for the case $a_k r_k(b_k t) = -|a_k|$.

Remark 5. The above result was proved for $b_m = 12^{m-1}$, $a_m = 6^{m-1}$ for $m = 1, 2, \ldots$ by McShane and Botts ([14], p. 113). Hobson essentially proved Theorem 6 ([8], pp. 410-411) in its full generality by utilizing a technique of Knopp.

Remark 6. A. S. Besicovitch has given an example of a continuous function which has no unilateral derivative (infinite or finite) at any point ([9], p. 172; [4], p. 39).

We now prove

THEOREM 7. *If* $a_m = o(1)$ *and* $\sum |a_m| = \infty$, *then for every* $\varepsilon > 0$, $(a, b) \subset [0, 1]$, *and real number* d, *there exists* $t_1, t_2 \in (a, b)$ *satisfying*

$$|[F(t_2)-F(t_1)]/[t_2-t_1]-d| < \varepsilon.$$

Proof. If $b_m = 2^{m-1}$, $a_m = o(1)$, and $\sum |a_m| = \infty$, then in every interval the Rademacher series $\sum_{m=1}^{\infty} a_m r_m(t)$ assumes every real number ([10], p. 234, Théorème 2). Assume now that $\sum_{m=1}^{\infty} a_m r_m(t_0) = d$ and choose $p/2^n \leqslant t_0 < (p+1)/2^n$. Then

$$\{F[(p+1)/2^n]-F[p/2^n]\}2^n = \sum_{m=1}^{n-1} a_m r_m(t_0) \to d \quad \text{as } n \to \infty.$$

The general case, for b_m/b_{m-1} even, is obtained by observing that series of the form (2) also assume in every interval every real number.

THEOREM 8. *If* $\sum |a_m b_m^{-1}| < \infty$ *and* $\sum a_m^2 = \infty$, *then* F *is continuous and has a finite derivative almost nowhere.*

Proof. If $b_m = 2^{m-1}$ and $\sum a_m^2 = \infty$, then the Rademacher series $\sum a_m r_m(t)$ diverges for almost every $t \in [0, 1)$ ([1], p. 54) and thus our result follows from Lemma 2.

The general case, when b_m/b_{m-1} is even, is obtained by generalizing Theorem 1.7.4 in [1], p. 54, to orthogonal series of the form (2).

THEOREM 9. *If* $\sum a_m^2 < \infty$, *then* F *is the primitive of a function* $f \in L^p, 0 < p < \infty$.

Proof. If $b_m = 2^{m-1}$ and $\sum a_m^2 < \infty$, then $\sum a_m r_m(t)$ converges almost everywhere and is the Fourier series of its sum $f \in L^p$, $0 < p < \infty$ ([18], pp. 212-213). Also by the Riesz-Fischer theorem ([18], p. 127)

$$\int_0^t \Big[f(x) - \sum_{m=1}^{n} a_m r_m(x)\Big] dx \to 0 \quad \text{as } n \to \infty,$$

i.e.,

$$F(t) = \sum_{m=1}^{\infty} a_m \int_0^t r_m(x)\, dx = \int_0^t \Big[\sum_{m=1}^{\infty} a_m r_m(x)\Big] dx.$$

The general case is obtained by generalizing Theorems (8.2) and (8.4) in [18], p. 212-213, to sieries of the form (2).

THEOREM 10. *If* $a_m \geqslant 0$, $\sum a_m = \infty$, *and* $\sum a_m b_m^{-1} < \infty$, *then* F *is a continuous function with a right-hand derivative of* $+\infty$ *and a left-hand derivative of* $-\infty$ *at all numbers of the form* $\{p/b_k\}$, *where* p *is any integer and* $k = 1, 2, \ldots$

Proof (cf. [7], p. 29). If $n > k$ and $b_{n+1}^{-1} \leqslant h < b_n^{-1}$, then

$$F(pb_k^{-1}+h) - F(pb_k^{-1}) \geqslant \sum_{m=1}^{n-1} a_m \int_{pb_k^{-1}}^{pb_k^{-1}+h} r_1(b_m t)\, dt$$

$$= \sum_{m=1}^{k-1} + \sum_{m=k}^{n-1} \geqslant -h \sum_{m=1}^{k-1} + h \sum_{m=k}^{n-1} a_m,$$

$$[F(pb_k^{-1}+h) - F(pb_k^{-1})]/h \geqslant -\sum_{m=1}^{k-1} a_m + \sum_{m=k}^{n-1} a_m \to +\infty \quad \text{as } h \to 0.$$

Similarly

$$[F(pb_k^{-1}-h) - F(pb_k^{-1})]/h \geqslant -\sum_{m=1}^{k-1} a_m + \sum_{m=k}^{n-1} a_m \to +\infty \quad \text{as } h \to 0.$$

COROLLARY 4. *If* $a_m \geqslant 0$, $\sum a_m^2 < \infty$, *and* $\sum a_m = \infty$, *then* F *is an absolutely continuous, everywhere oscillating function with a proper minimum at all numbers of the form* $\{p/b_k\}$.

Remark 7. A function may be everywhere oscillating and have a finite derivative at every point ([3], p. 61; [8], p. 412).

THEOREM 11. *If* $\sum |a_m| < \infty$, *then* $F \in \mathrm{Lip}\,1$ *and has a right- and left-hand derivative everywhere.*

Proof. If $\sum |a_m| < \infty$, then series (2) is uniformly absolutely convergent and hence its sum $f(t)$ must have right- and left-hand limits everywhere.

COROLLARY 5. *If* $b_m = 2^{m-1}$, $\sum |a_m| < \infty$, *and*

$$a_k \neq \sum_{m=k+1}^{\infty} a_m \quad \textit{for } k = 1, 2, \ldots,$$

then F *is continuous, has a symmetric derivative* (defined in [4], p. 34) *everywhere, but is not differentiable at dyadic rationals.*

Proof. If a function has a right-and legt-hand derivative at a point, then it has a symmetric derivative there.

Also, since $r_m(p2^{-k}+t) = r_m(t)$ if $m \geqslant k+1$, and $r_k(p2^{-k}+t) = -r_k(t)$, we have

$$(14) \quad f(p2^{-k}+t) - f(p2^{-k}-t) = \sum_{m=1}^{k-1} a_m [r_m(p2^{-k}+t) - r_m(p2^{-k}-t)] + \\ + a_k[-r_k(t) + r_k(-t)] + \sum_{m=k+1}^{\infty} a_m [r_m(t) - r_m(-t)].$$

Hence for p odd, it follows that (14) approaches $-2a_k + 2\sum_{m=k+1}^{\infty} a_m$ as $t \to 0+$, and so by our hypothesis $f(t)$ must have unequal right- and left-hand limits at all dyadic rationals. But this implies $F(t)$ will have different right- and left-hand derivatives at these points.

Remark 8. Corollary 5 for the special case $a_m = (2/3)^m$ for $m = 1, 2, \ldots$ was recently proved by Mukhopadhyay [15]. Mukhopadhyay's theorem can also be obtained by considering the primitive of the function

$$g(t) = \sum_{n \in E(t)} 2^{-n},$$

where $E(t) = \{n\colon t_n \leqslant t\}$ and $\{t_n\}$ is a dense denumerable set; thus making $g(t)$ monotone, bounded, and having at each t_n an essential jump discontinuity.

We now prove that no condition weaker than Lip 1 implies differentiability anywhere.

THEOREM 12. *If $\omega(h)\uparrow\infty$ arbitrarily slowly as $h\downarrow 0$, then there exists a continuous function F, with a finite derivative nowhere, such that*

$$\omega(F, h) = O[h\omega(h)],$$

where $\omega(F, h)$ denotes the modulus of continuity of F.

Proof. Choose a non-null sequence $\{a_m\}$ satisfying

(i) a_m is either 0 or 1 for $m = 1, 2, \ldots,$

(ii) $\sum_{m=1}^{n} a_m = O[\omega(2^{-n})]$.

Then, setting $b_m = 2^{m-1}$ in (2), we have by Lemma 3

$$|F(t+h)-F(t)| = O\Big[h\omega(2^{-n})+\sum_{m=n+1}^{\infty} 2^{-n}\Big] = O[h\omega(h)].$$

But since $a_m \neq o(1)$, F has a finite derivative nowhere by Theorem 3 (or Theorem 4).

Remark 9. The last result was proved by Kahane [11] by a similar method. In a previous paper of the author ([12], Theorem 3.7), by use of trigonometric series, we constructed a function F, as in Theorem 12, which was differentiable almost nowhere. However, by selecting $\{\varepsilon_m\}$ in our previous paper as $\{a_m\}$ in Theorem 12 we could have obtained differentiability nowhere by utilizing a result of Freud ([6], p. 261) on lacunary trigonometric series.

Remark 10. For other interesting properties of Rademacher series the reader is referred to [13] and its references.

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Reçu par la Rédaction le 24. 2. 1968