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DEGREE OF CONVERGENCE OF SOME SEQUENCES IN THE CONFORMAL MAPPING THEORY

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Introduction. The purpose of this paper is to give estimates for the degree of convergence of some number sequences converging to the logarithmic capacity or to the hyperbolic capacity of a compact plane set E as well as estimates for the degree of convergence of some sequences of functions converging to the Green function $G(z)$ of the unbounded component D_∞ of $C\setminus E$ with a pole at ∞ or to a function closely related to the function mapping conformally a doubly connected domain onto an annulus. In the case of hyperbolic capacity we assume that E is a compact subset of the unit disc $\{|z|<1\}$. We prove that if each component of E has the ordinary diameter $\geqslant 2r$, r being a fixed positive number, then the degrees of convergence of all the sequences are estimated by expressions of the form $O\big(\log(n+1)/n\big)$. In the case of sequences of functions the estimating expressions depend also on z.

Our results generalize results due to Kleiner [2] and [3] and Pommerenke [6] and [7], who considered only the case of logarithmic capacity of a connected set. Also they used different techniques of proof. Our method of proof is based on the continuity behaviour of the Green function $G(z)$ if a point z tends to a point in E (see Lemma 1). We prove it by means of an inequality due to Leja [4, 5], used by him for deriving a very useful lemma on polynomials.

1. A continuity property of the Green function. Let E be a bounded closed set in the complex plane C. Denote by D_∞ the unbounded component of $C\setminus E$. For each finite $z\in C$ define $L(z)=L(z,E)$ by

$$(1.1)\qquad L(z)=\sup\sqrt[n]{|P_n(z)|},$$

the sup being taken over all polynomials $P_n(z)=a_nz^n+\ldots+a_0$, $n\geqslant 1$, such that $|P_n(z)|\leqslant 1$ on E. It is known [5, 10] that $L(z)=1$ in $C\setminus D_\infty$, $L(z)>1$ in D_∞. Moreover, if the logarithmic capacity $d(E)$ of E is positive, then the restriction of $G(z)=\log L(z)$ to D_∞ is the generalized Green function of D_∞ with a pole at ∞. We shall prove the following

LEMMA 1. *Assume that there exists a real number $r > 0$ such that if F is any component of E, then $\sup_{z,\zeta\in E}|z-\zeta| \geqslant 2r$. Then for each $\delta > 0$ and for all $z\in C$ such that* $\operatorname{dist}(z, E) = \min_{\zeta\in E}|z-\zeta| \leqslant \delta$ *we have*

$$G(z) \leqslant \left(\pi+\sqrt{\frac{\delta}{r}}\right)\sqrt{\frac{\delta}{r}}. \tag{1.2}$$

Proof. In [4] (see also [5], p. 272-273) we find the inequality

$$\sqrt[n]{|P_n(z)|} \leqslant [2(n+1)\max_{\zeta\in E}|P_n(\zeta)|]^{1/n}e^{\lambda_n}, \quad \text{if} \quad \operatorname{dist}(z, E) < \delta, \tag{1.3}$$

$P_n(z)$ being any polynomial of degree $\leqslant n$ and

$$\lambda_n = \frac{1}{n}\sum_{k=1}^{n}\log\frac{\alpha^2+k^2/n^2}{k^2/n^2}, \quad \alpha^2 = \frac{\delta}{r}.$$

By (1.1) and (1.3) we have

$$G(z) \leqslant \log(1+\alpha^2)+2\alpha\operatorname{arc\,tg}\frac{1}{\alpha}, \quad \text{if} \quad \operatorname{dist}(z, E) < \delta,$$

because

$$\lambda_n \to \int_0^1 \log\frac{\alpha^2+x^2}{x^2}\,dx = \log(1+\alpha^2)+2\alpha\operatorname{arc\,tg}\frac{1}{\alpha}.$$

But $\log(1+\alpha^2) \leqslant \alpha^2$ and $\operatorname{arc}(\operatorname{tg} 1/\alpha) \leqslant \pi/2$, so the proof is concluded.

Remark. Let $E_\delta = \{z: G(z) = \log(1+\delta)\}$, $\delta > 0$. Inequality (1.2) implies that there exists a positive constant $K = K(E)$ such that $\operatorname{dist}(E_\delta, E) \geqslant K\delta^2$ for $\delta > 0$. This may be used for deriving the following inequality: $|P_n'(z)| \leqslant Men^2/K$ on E, P_n being a polynomial of degree $\leqslant n$ such that $|P_n(z)| \leqslant M$ on E (see [9]).

2. Degree of convergence in approximations of the logarithmic capacity and of the Green function. Given any system $\zeta^{(n)} = \{\zeta_0, \ldots, \zeta_n\}$ of $n+1$ points of the complex plane C we put

$$V(\zeta^{(n)}) = \prod_{0\leqslant i<k\leqslant n}|\zeta_i-\zeta_k|, \tag{2.1}$$

$$\Delta^{(j)}(\zeta^{(n)}) = \prod_{k=0(k\neq j)}^{n}(\zeta_j-\zeta_k), \quad j = 0, \ldots, n,$$

$$\omega(z, \zeta^{(n)}) = \prod_{k=0}^{n}(z-\zeta_k), \tag{2.2}$$

$$L^{(j)}(z, \zeta^{(n)}) = \frac{\omega(z, \zeta^{(n)})}{(z-\zeta_j)\Delta^{(j)}(\zeta^{(n)})}, \quad j = 0, \ldots, n, \tag{2.3}$$

the last definition being valid only under the assumption that $\zeta_j \neq \zeta_k$ for $j \neq k$. Let E be a bounded closed infinite subset of C. Denote by

$$\eta^{(n)} = \{\eta_{0n}, \eta_{1n}, \ldots, \eta_{nn}\}, \qquad n = 1, 2, \ldots, \tag{2.4}$$

a fixed system of $n+1$ points of E such that $V(\zeta^{(n)}) \leqslant V(\eta^{(n)})$ for every $\zeta^{(n)} \subset E$. We shall always assume that the points of the extremal system (2.4) are numbered in such a way that

$$|\Delta^{(0)}(\eta^{(n)})| \leqslant |\Delta^{(j)}(\eta^{(n)})|, \quad j = 0, \ldots, n. \tag{2.5}$$

Let d_n, δ_n, K_n, be defined by

$$\begin{gathered} d_n = [V(\eta^{(n)})]^{2/n(n+1)}, \qquad \delta_n = \sqrt[n]{|\Delta^{(0)}(\eta^{(n)})|}, \\ K_{n+1} = [\max_{z\epsilon E} |\omega(z, \eta^{(n)})|]^{1/(n+1)}, \qquad n \geqslant 1, \end{gathered} \tag{2.6}$$

and let two further sequences r_n and $\mathring{r}_n$ be defined by

$$\begin{gathered} r_{n+1}^{n+1} = \min_{\zeta^{(n)} \subset C} [\max_{z\epsilon E} |\omega(z, \zeta^{(n)})|], \\ \mathring{r}_{n+1}^{n+1} = \min_{\zeta^{(n)} \subset E} [\max_{z\epsilon E} |\omega(z, \zeta^{(n)})|], \qquad n \geqslant 1. \end{gathered} \tag{2.7}$$

Fekete [1] proved that the sequence $\{d_n\}$ is convergent. Its limit $d = \lim d_n$ is called the *transfinite diameter of E* or the *logarithmic capacity of E*. The inequalities

$$\begin{aligned} d \leqslant r_n \leqslant \mathring{r}_n \leqslant K_n \leqslant \left[\frac{V(\eta^{(n)})}{V(\eta^{(n-1)})}\right]^{1/n} \leqslant \delta_n \\ \leqslant d_n \leqslant [\delta_1 \delta_2^2 \ldots \delta_n^n)]^{2/n(n+1)}, \qquad n \geqslant 1, \end{aligned} \tag{2.8}$$

may be proved by standard reasonings (comp. [5], [12], [7]). So all the sequences in (2.8) are convergent to the same limit $d = d(E)$.

Suppose $d(E) > 0$ and let $L(z)$ be defined by (1.1). Then $\lim_{z\to\infty} L(z)/|z| = 1/d$ and $G(z) = \log L(z)$ restricted to D_∞ is the generalized Green's function of D_∞. It is also known [5] that the sequence $L_n(z) = \sqrt[n]{|L^{(0)}(z, \eta^{(n)})|}$, $n = 1, 2, \ldots$, is convergent to $L(z)$, $z \epsilon D_\infty$. Moreover, if D_∞ is simply connected, then

$$e^{i\theta_n} \sqrt[n]{L^{(0)}(z, \eta^{(n)})} = \frac{z}{\delta_n} + a_{0n} + \frac{a_{1n}}{z} + \ldots, \qquad n \geqslant 1,$$

where θ_n are suitably chosen real numbers, is convergent in D_∞ to the univalent conformal mapping of D_∞ onto $\{|w| > 1\}$. We shall prove the following

THEOREM 1. *Let E be a bounded closed subset of C such that the function $L(z)$, given by (1.1), is continuous in C. Let $\Omega(\delta)$ be a positive continuous function defined for $\delta > 0$ such that $\lim_{\delta\to 0}\Omega(\delta) = 0$ and $\log L(z) = G(z) \leqslant \Omega(\delta)$ if $\operatorname{dist}(z, E) < \delta$. Then for $n = 1, \ldots$ we have*

(a) $$0 \leqslant G(z) - \log L_n(z)$$
$$\leqslant \frac{3}{n}\log[R(n+1)] + \Omega\left(\frac{1}{n^2}\right), \text{ if } \operatorname{dist}(z, E) \geqslant \frac{1}{n^2}, \qquad z \in D_\infty,$$

(b) $$0 \leqslant \log(\delta_n/d) \leqslant \frac{3}{n}\log[R(n+1)] + \Omega\left(\frac{1}{n^2}\right),$$

where $R = [\sup_{z,\zeta\in E}|z-\zeta| + 2]^{1/3}$.

Proof. By (2.3) and by (2.5) we have

$$|L^{(0)}(z, \eta^{(n)})| = |L^{(j)}(z, \eta^{(n)})| \frac{|\Delta^{(j)}(\eta^{(n)})|}{|\Delta^{(0)}(\eta^{(n)})|} \frac{|z-\eta_{jn}|}{|z-\eta_{0n}|}$$
$$\geqslant |L^{(j)}(z, \eta^{(n)})| \frac{r(z)}{R(z)}, \qquad j = 0, \ldots, n,$$

where $r(z) = \operatorname{dist}(z, E)$, $R(z) = \max_{\zeta\in E}|z-\zeta|$. The identity

$$1 \equiv \sum_{j=0}^{n} L^{(j)}(z, \eta^{(n)})$$

implies that

$$\max_{0\leqslant j\leqslant n}|L^{(j)}(z, \eta^{(n)})| \geqslant \frac{1}{n+1} \text{ in } C.$$

Therefore

$$L(z)/L_n(z) \leqslant [(n+1)R(z)/r(z)]^{1/n} L(z), \qquad z \in D_\infty.$$

The function $U_n(z) = \log[L(z)/L_n(z)]$, $U_n(\infty) = \log(\delta_n/d)$, is harmonic in D_∞. One easily checks that $|L^{(0)}(z, \eta^{(n)})| \leqslant 1$ on E. So by (1.1) we have $L_n(z) \leqslant L(z)$ in C. Therefore $U_n(z) \geqslant 0$ in C. Then

$$0 \leqslant U_n(z) \leqslant \frac{1}{n}\log\frac{(n+1)R(z)}{r(z)} + \Omega(\delta), \tag{2.9}$$

if $r(z) = \operatorname{dist}(z, E) \leqslant \delta$, $z \in D_\infty$.

Let D_n be a subset of D_∞ defined by

$$D_n = \{z \in D_\infty : \operatorname{dist}(z, E) \geqslant 1/n^2\}.$$

The D_n is a closed subset of D_∞, $D_n \subset D_{n+1}$, and $D = \lim_{n\to\infty} D_n$. By the maximum property of harmonic functions and by (2.9) we get

$$0 \leqslant U_n(z) \leqslant \frac{3}{n}\log[R(n+1)] + \Omega\left(\frac{1}{n^2}\right) \quad \text{for} \quad z \in D_n, \tag{2.10}$$

where $R = [\sup_{z,\zeta\in E}|z-\zeta|+2]^{1/3}$. The proof of (a) is achieved. Inequality (b) is an immediate consequence of (a).

Combining Lemma 1 and Theorem 1 and using the last inequality of (2.8) we get the following

THEOREM 2. *If E is a bounded closed set each component of which has the ordinary diameter $\geqslant 2r$, r being a fixed positive number, then for $n = 1, 2, \ldots$ we have*

$$0 \leqslant G(z) - \log L_n(z) \leqslant \frac{3}{n}\log[R(n+1)] + \left(\pi + \frac{1}{\sqrt{rn}}\right)\frac{1}{\sqrt{rn}}, \tag{a}$$

if $\operatorname{dist}(z, E) \geqslant \frac{1}{n^2}$, $\quad z \in D_\infty$,

$$0 \leqslant \log(\delta_n/d) \leqslant \frac{3}{n}\log[M(n+1)], \tag{b}$$

$$0 \leqslant \log(d_n/d) \leqslant \frac{6}{n}\log[M(n+1)], \tag{c}$$

where $R = [\sup_{z,\zeta\in E}|z-\zeta|+2]^{1/3}$ and $M = R\exp[(\pi+1/\sqrt{r})1/\sqrt{r}]$.

Let us now prove Lemma 2 which is a version of the Harnack's theorem.

LEMMA 2. *Let D be a domain in the closed plane $\bar{C}$. Let $\{u_n(z)\}$ be a sequence of non-negative harmonic functions in D. If $u_n(z_0) \to 0$ for a fixed point $z_0 \in D$, then there exists a positive function $K(z)$ continuous in D such that*

$$0 \leqslant u_n(z) \leqslant K(z)u_n(z_0), \quad z \in D, \quad n = 1, 2, \ldots \tag{2.11}$$

Proof. Let $\{D_\nu\}$ be a sequence of domains with the following properties:

1° $z_0 \in D_\nu$, $D_\nu \subset D_{\nu+1}$, $\nu = 1, 2, \ldots$ and $D = \lim_{\nu\to\infty} D_\nu$;

2° the boundary Γ_ν of D_ν is a sum of finitely many disjoint analytic closed curves oriented positively with respect to D_ν.

Let $G_\nu(z, \zeta)$ denote the Green function of D_ν with a pole at ζ. If n_ζ is the normal to Γ_ν at $\zeta \in \Gamma_\nu$, directed into the interior of D_ν, then by the well known Green's formula we have

$$u_n(z) = \frac{1}{2\pi}\int\limits_{\Gamma_\nu} u_n(\zeta)\frac{\partial G_\nu(\zeta, z)}{\partial n_\zeta}\, ds_\zeta, \qquad z \in D_\nu.$$

The function $\partial G_\nu(\zeta, z)/\partial n_\zeta$ is positive and continuous for $\zeta \in \Gamma_\nu$, $z \in \overline{D}_{\nu-1}$. Therefore the function

$$K_\nu(z) = \sup_{\zeta\in\Gamma_\nu}\left[\frac{\partial G_\nu(\zeta, z)}{\partial n_\zeta}\Big/\frac{\partial G_\nu(\zeta, z_0)}{\partial n_\zeta}\right], \qquad z \in D_{\nu-1},$$

is continuous and positive in $D_{\nu-1}$, and

$$u_n(z) \leqslant K_\nu(z)\frac{1}{2\pi}\int\limits_{\Gamma_\nu} u_n(\zeta)\frac{\partial G_\nu(\zeta, z_0)}{\partial n_\zeta}\, ds_\zeta = K_\nu(z)\, u_n(z_0), \qquad z \in D_{\nu-1}.$$

Define in D a new function $K^*(z)$ by

$$K^*(z) = \inf_{l\geqslant\nu+1} K_l(z), \quad \text{if} \quad z \in D_\nu.$$

The function $K^*(z)$ is finite and upper-semicontinuous in D. So, by a theorem of Baire, there exists a continuous function $K(z) \geqslant K^*(z)$, $z \in D$, such that (2.11) holds, q.e.d.

We could also prove Lemma 2 by using the Poisson formula instead of the Green formula. It follows from Lemma 2 that the degree of convergence of $u_n(z)$ on an arbitrary compact subset of D is uniformly the same as at a fixed point $z_0 \in D$.

In particular, if $Q_n(z) = z^n + \ldots$, $\quad n = 1, 2, \ldots$, is a sequence of polynomials such that

$$M_n = [\max_{z\in E}|Q_n(z)|]^{1/n} \to d = d(E)$$

and all zeros of the polynomials are contained in $C \setminus D_\infty$, then

$$(2.12) \qquad 0 \leqslant G(z) - \log\frac{|Q_n(z)|^{1/n}}{M_n} \leqslant K(z)\log(M_n/d), \qquad z \in D_\infty,$$

$K(z)$ being a continuous function in D_∞.

The numbers r_n^n, $\mathring{r}_n^n$, K_n and δ_n^n are maxima on E of moduli of suitably chosen polynomials $Q_n(z)$. In all these cases we may get estimates of the form (2.12) in D_∞ (or in the complement of the convex envelope of E in the case of r_n). We know [6] that if E is convex, then

$$0 \leqslant \log(r_n/d) \leqslant \frac{1}{n}\log 2.$$

So, if $T_n(z) = z^n + \ldots$ is the Tchebycheff polynomial of a convex compact set E, then

$$0 \leqslant G(z) - \log \frac{|T_n(z)|^{1/n}}{r_n} \leqslant K(z)\frac{1}{n}, \qquad z \in D_\infty.$$

The estimate $\log(r_n/d) \leqslant K/n$, $K = \text{const}$, is also valid if E is a sum of finitely many disjoint analytic Jordan curves [11].

3. Degree of convergence in approximations of the hyperbolic capacity and of a conformal mapping of a doubly connected domain. Let E be a closed infinite subset of the unit disc $\{|z| < 1\}$. Let $E^* = \{z \in C: (1/\bar{z}) \in E\}$, where $\bar{z}$ is the conjugate of z. Denote by D the component of $C \setminus (E \cup E^*)$ containing the circle $\{|z| = 1\}$. Let

$$w(z, \zeta) = \frac{z-\zeta}{1-z\bar{\zeta}}$$

and put

$$V(\zeta^{(n)}) = \prod_{0 \leqslant i < k \leqslant n} |w(\zeta_i, \zeta_k)|,$$

where $\zeta^{(n)} = \{\zeta_0, \ldots, \zeta_n\}$ is a system of $n+1$ points of E. Let

$$\eta^{(n)} = \{\eta_{0n}, \eta_{1n}, \ldots, \eta_{nn}\} \tag{3.1}$$

be a fixed extremal system of E with respect to $|w(z, \zeta)|$, i.e. $V(\zeta^{(n)}) \leqslant V(\eta^{(n)})$ for $\zeta^{(n)} \subset E$. Let the points of $\eta^{(n)}$ be numbered in such a way that

$$\Delta^{(0)}(\eta^{(n)}) \leqslant \Delta^{(i)}(\eta^{(n)}) = \prod_{k=0 (k \neq i)}^{n} |w(\eta_{in}, \eta_{kn})|, \qquad i = 0, \ldots, n. \tag{3.2}$$

According to [12] and [8] the sequences

$$d_n = [V(\eta^{(n)})]^{2/n(n+1)} \quad \text{and} \quad \delta_n = \sqrt[n]{\Delta^{(0)}(\eta^{(n)})}, \qquad n \geqslant 1,$$

are convergent to the same limit $d_- = d_-(E)$ which is called the *hyperbolic capacity of E*. Moreover [12],

$$\log 1/d_- = \inf_{\mu} \int_E \int_E \log \frac{1}{|w(z, \zeta)|} d\mu(z) d\mu(\zeta), \tag{3.3}$$

μ being a positive mass distribution on E of total mass 1.

Let r_n, $\mathring{r}_n$ and K_n be defined by

$$r_n = \min_{|z_\nu| < 1} \Big[\max_{z \in E} \prod_{\nu=0}^{n} |w(z, z_\nu)|\Big]^{1/n}, \qquad \mathring{r}_n = \min_{z_\nu \in E} \Big[\max_{z \in E} \prod_{\nu=1}^{n} |w(z, z_\nu)|\Big]^{1/n},$$

$$K_n = \Big[\max_{z \in E} \prod_{\nu=0}^{n-1} |w(z, \eta_{\nu, n-1})|\Big]^{1/n}.$$

One may prove that the present sequences $r_n, \mathring{r}_n, K_n$, $[V(\eta^{(n)})/V(\eta^{(n-1)})]^{1/n}$, δ_n and d_n also satisfy the inequalities (2.8). So all the sequences are convergent to the hyperbolic capacity of E, and the degree of convergence of each of them may be estimated by the degree of convergence of $\{d_n\}$ or of $\{\delta_n\}$.

Let $d_-(E) > 0$ and let

$$g_n(z) = \frac{1}{\delta_n}\Big[\prod_{\nu=1}^{n} w(z, \eta^{(n)})\Big]^{1/n}.$$

Then, according to [8], there exists a multivalued function $g(z)$ analytic in $D_1 = \{z \in D: |z| \leqslant 1\}$ such that $\{|g_n(z)|\}$ is convergent to $|g(z)|$ in D_1. The function $g(z)$ satisfies the inequalities $1 \leqslant |g(z)| \leqslant 1/d_-$ in the interior of D_1 and $|g(z)| = 1/d_-$ on $\{|z| = 1\}$. Moreover, if each point of E is contained in E along with a continuum not reduced to a point, then $\lim|g(z)| = 1$ as z tends from D_1 to a point in E. In the sequel we shall assume that $|g(z)|$ is defined also in E by putting $|g(z)| = 1$ in E. By the reflection principle $|g_n(1/\bar{z})| = 1/[\delta_n^2 |g_n(z)|]$ for $z \in D$. So we may assume that $g(z)$ is defined in D, $|g(z)| = \lim|g_n(z)|$, $z \in D$. Obviously the function $\log|g(z)|$ is harmonic in D. If E is connected, the functions $g_n(z)$ and $g(z)$ may be assumed to be single valued in D and the function g transforms the doubly connected domain $\mathring{D}_1$ onto the annulus $1 < |w| < 1/d_-$.

By the way, let us observe that an elementary proof of the last fact given in [8] is not complete because the author assumes without proof that if E is connected, then $|g(z)|$ takes in D_1 each value of the inteval $(1, 1/d_-)$. However, the gap may be filled up in many ways.

THEOREM 3. *If E is a closed subset of $\{|z| < 1\}$ each component of which has the ordinary diameter $\geqslant 2r = \text{const} > 0$, then there exist positive numbers K and K_1 such that*

(a) $$0 \leqslant \log(d_n/d_-) \leqslant K\frac{\log(n+1)}{n}, \qquad n = 1, 2, \ldots,$$

(b) $$0 \leqslant \log[|g(z)|/|g_n(z)|] \leqslant K_1\frac{\log(n+1)}{n},$$

if $z \in D_1$ and $\operatorname{dist}(z, E) \geqslant 1/n^2$.

Proof. Take $L(z) = \exp G(z)$, where $G(z)$ is the Green function of the unbounded component of $C\setminus E$ with pole at ∞. Since $L(z) > 1$ on $\{|z| = 1\}$ so there is a number $m > 0$ such that $L^m(z) > 1/d_-(E)$ on $\{|z| = 1\}$. By the maximum principle $|g(z)| \leqslant L^m(z)$ in D_1. Hence by (1.2) we have

$$(3.4)\quad \log|g(z)| \leqslant \Omega(\delta) = m\left(\pi+\sqrt{\frac{\delta}{r}}\right)\sqrt{\frac{\delta}{r}}, \text{ if } z\in D_1 \text{ and } \operatorname{dist}(z, E) < \delta.$$

We shall assume that $\delta > 0$ is sufficiently small so that

$$\Omega(\delta) < \min_{|z|=1,\zeta\in E} |z-\zeta|.$$

The set $\Gamma_\delta = \{z\in \bar{D}_1: \log|g(z)| \leqslant \Omega(\delta)\}$ contains the set $E_\delta = \{z: \operatorname{dist}(z, E) \leqslant \delta, z\in \bar{D}_1\}$. One may easily prove that

$$(3.5)\qquad d_-(E_\delta) \leqslant d_-(\Gamma_\delta) = d_-(E)e^{\Omega(\delta)}.$$

We shall now prove (a) by using a slight modification of a reasoning due to M. Tsuji ([12], p. 95-96). Take $n+1$ distinct points z_i, $i = 0, 1, \ldots, n$, on E and put $\Delta_i = \{z: |z-z_i| = 1/n^2\}$. Assume n is so large that

$$\Omega\left(\frac{1}{n^2}\right) < \min_{|z|=1,\zeta\in E} |z-\zeta|.$$

Let σ_i be the mass $1/(n+1)$ spread with constant density on Δ_i. Put $\sigma = \sigma_0+\ldots+\sigma_n$. Then by (3.3) and (3.5) we have

$$\log\frac{1}{d_-(E)} - \Omega(\delta) \leqslant \log\frac{1}{d_-(E_\delta)} \leqslant \int_{E_\delta}\int_{E_\delta} \log\frac{1}{|w(z,\zeta)|}\, d\sigma(z)\, d\sigma(\zeta)$$
$$= \sum_{j=0}^{n}\int_{\Delta_j} d\sigma_j(z) \sum_{i=0}^{n}\int_{\Delta_i} \log\frac{1}{|w(z,\zeta)|}\, d\sigma_i(\zeta),$$

where $\delta = 1/n^2$. Since $\log 1/|w(z, \zeta)|$ is a superharmonic function of ζ, we have

$$\int_{\Delta_i} \log\frac{1}{|w(z,\zeta)|}\, d\sigma_i(\zeta) \leqslant \frac{1}{n+1}\log\frac{1}{|w(z, z_i)|},$$

so that

$$\log\frac{1}{d_-(E_\delta)} \leqslant \frac{1}{n+1}\sum_{i,j=0}^{n}\int_{\Delta_j} \log\frac{1}{|w(z, z_i)|}\, d\sigma_j(z).$$

If $i \neq j$, then

$$\int_{\Delta_j} \log\frac{1}{|w(z, z_i)|}\, d\sigma_j(z) \leqslant \frac{1}{n+1}\log\frac{1}{|w(z_j, z_i)|}.$$

If $i = j$, then

$$\int_{\Delta_i} \log \frac{1}{|w(z, z_i)|} d\sigma_i(z) \leqslant \frac{2}{n+1} \log(n+1) + \frac{1}{n+1} \log(1-a^2), \qquad a = \sup_{z \in E} |z|.$$

Therefore

$$\log \frac{1}{d_-(E)} - \Omega(\delta) \leqslant \frac{n}{n+1} \log \frac{1}{d_n} + \frac{2}{n+1} \log(n+1) + \frac{1}{n+1} \log(1-a^2).$$

Since $\delta = 1/n^2$, this inequality implies (a).

Remark. By an analogous reasoning, using the continuity property (1.2) of the Green function $G(z)$, we could give a new proof of the corresponding estimates for the transfinite diameter.

To prove (b) we shall use the following inequality [8]

$$|g_n(z)|^n \geqslant \frac{(1-a^2) r(z)}{2R(z)(n+1)}, \qquad z \in D_1, n = 1, 2, \ldots, \tag{3.6}$$

where $r(z) = \operatorname{dist}(z, E)$, $R(z) = \sup_{\zeta \in E} |z - \zeta|$, $a = \max_{z \in E} |z|$.

Let Γ_n be a subset of D_1 defined by $\Gamma_n = \{z \in D_1 \colon \operatorname{dist}(z, E) = 1/n^2\}$. By (3.4) and (3.6) we have

$$\begin{aligned} 0 \leqslant \log \frac{|g(z)|}{|g_n(z)|} &\leqslant \frac{1}{n} \log \frac{4(n+1)^3}{1-a^2} + \Omega\left(\frac{1}{n^2}\right) \\ &\leqslant K_2 \frac{\log(n+1)}{n}, \qquad K_2 = \text{const}, z \in \Gamma_n. \end{aligned} \tag{3.7}$$

If $|z| = 1$, then $0 \leqslant \log[|g(z)|/|g_n(z)|] = \log(\delta_n/d_-) \leqslant \log(d_n/d_-)$. Now (b) follows from (3.7), (a) and from the maximum principle.

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