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ON A PROBLEM OF SIKORSKI IN THE SET REPRESENTABILITY OF BOOLEAN ALGEBRAS

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Introduction. In dealing with questions of set representability of Boolean algebras, Sikorski has defined classes CSR, $K_{\mathfrak{n}}$ and $\mathfrak{n}$SR, of Boolean algebras, where $\mathfrak{n}$ is an infinite cardinal. He has asked, in [8] and [9], whether $K_{\mathfrak{n}}$ is distinct from CSR in case $\mathfrak{n}$ is uncountable. Sierpiński showed, in [6] and [7], that $K_{\aleph_0}$ is distinct from CSR. In this paper, we construct for each infinite $\mathfrak{n}$ a Boolean algebra belonging to $K_{\mathfrak{n}}$ but not to CSR. It will follow that CSR is properly contained in $K_{\mathfrak{n}}$ and that $K_{\mathfrak{n}}$ is properly contained in $\mathfrak{n}$SR. No Boolean algebra in $K_{\mathfrak{n}}-$CSR can have cardinality less than $\mathfrak{n}^+$; and we show that if $\mathfrak{n}^+ = 2^{\mathfrak{n}}$, then there is a Boolean algebra of cardinality $\mathfrak{n}^+$ which is in $K_{\mathfrak{n}}-$CSR. We also show that no Boolean algebra in $K_{\mathfrak{n}}-$CSR can be $\aleph_0$ complete.

Preliminaries. A Boolean algebra $\mathscr{A}$ is called *completely set-representable* if there is a set Y and an isomorphism h from $\mathscr{A}$ into the field of all subsets of Y which preserves all joins and meets that exist in $\mathscr{A}$. Similarly, $\mathscr{A}$ is called $\mathfrak{n}$-*set-representable* if such Y and h exist with h preserving those joins $\sum\limits_{t\in T} a_t$ and meets $\prod\limits_{t\in T} a_t$ of $\mathscr{A}$ for which T has cardinality at most $\mathfrak{n}$. CSR and $\mathfrak{n}$SR denote the classes of all Boolean algebras which are completely set-representable and $\mathfrak{n}$-set-representable, respectively.

We have the following equivalent formulations of CSR and $\mathfrak{n}$SR. Proofs can all be found in [10].

$\mathscr{A} \in$ CSR if and only if $\mathscr{A}$ is atomic. This result is due to Lindenbaum and Tarski and is proved in [11]. Also $\mathscr{A} \in$ CSR if and only if, in the Stone space of $\mathscr{A}$, the union of all nowhere dense sets is nowhere dense. A subset Z of a topological space is called $\mathfrak{n}$-*nowhere dense* if Z is a subset of a nowhere dense set X which is the intersection of at most $\mathfrak{n}$ open and closed sets. Now $\mathscr{A} \in \mathfrak{n}$SR if and only if, in the Stone space of $\mathscr{A}$, the union of all the $\mathfrak{n}$-nowhere dense sets is a boundary set (i. e. it has empty interior).

The intermediate class $K_{\mathfrak{n}}$ is defined by the following condition: $\mathscr{A} \in K_{\mathfrak{n}}$ if, in the Stone space of $\mathscr{A}$, the union of all $\mathfrak{n}$-nowhere dense sets is nowhere dense. To obtain an algebraic characterization of $K_{\mathfrak{n}}$ we need only compare its definition with the topological version of weak $(\mathfrak{m}, \mathfrak{n})$-distributivity. A set $\{a_{t,s}: t \in T, s \in S\}$ is called $(\mathfrak{m}, \mathfrak{n})$-*indexed* if T has cardinality at most $\mathfrak{m}$ and S has cardinality at most $\mathfrak{n}$. Let fin S denote the set of all finite, non-void subsets of S, and let $(\text{fin}\, S)^T$ denote the set of all functions from T into fin S. A Boolean algebra $\mathscr{A}$ is called *weakly* $(\mathfrak{m}, \mathfrak{n})$-*distributive* provided

$$\prod_{t \in T} \sum_{s \in S} a_{t,s} = \sum_{\Phi \in (\text{fin}\, S)^T} \prod_{t \in T} \sum_{s \in \Phi(t)} a_{t,s}$$

for every $(\mathfrak{m}, \mathfrak{n})$-indexed set $\{a_{t,s}: t \in T, s \in S\}$ of elements of $\mathscr{A}$ such that the meets $\prod_{t \in T} \sum_{s \in \Phi(t)} a_{t,s}$ exist, all the joins $\sum_{s \in S} a_{t,s}$ exist and the meet $\prod_{t \in T} \sum_{s \in S} a_{t,s}$ exists.

The following theorem is proved in [10]:

A Boolean algebra $\mathscr{A}$ *is weakly* $(\mathfrak{m}, \mathfrak{n})$-*distributive if and only if, in the Stone space of* $\mathscr{A}$, *the union of at most* $\mathfrak{m}$ $\mathfrak{n}$-*nowhere dense sets is nowhere dense.*

We see that $\mathscr{A} \in K_{\mathfrak{n}}$ if and only if, for every cardinal number $\mathfrak{m}$, $\mathscr{A}$ is weakly $(\mathfrak{m}, \mathfrak{n})$-distributive.

The proof that a Boolean algebra $\mathscr{A}$ belongs to $K_{\mathfrak{n}}$ is actually carried out by proving that $\mathscr{A}$ belongs to a smaller class $C_{\mathfrak{n}}$. To define $C_{\mathfrak{n}}$, a join $a = \sum_{t \in T} a_t$ is called an $\mathfrak{n}$-*join* if the cardinality of T is at most $\mathfrak{n}$. Now $\mathscr{A} \in C_{\mathfrak{n}}$ if every $\mathfrak{n}$-join which exists in $\mathscr{A}$ is essentially finite (i. e. there is a finite set $F \subseteq T$ such that $a = \sum_{t \in F} a_t$). It is easily seen, either algebraically or topologically, that $C_{\mathfrak{n}} \subseteq K_{\mathfrak{n}}$.

Example 1. There is a Boolean algebra $\mathscr{C}$ which is in $K_{\mathfrak{n}} - \text{CSR}$. The following construction is a combination of two simpler ones, and is due to D. Monk. First we note that for every Boolean algebra $\mathscr{A}$ there is a Boolean algebra $\mathscr{B}$ such that $\mathscr{A}$ is a subalgebra of $\mathscr{B}$ and no atom of $\mathscr{A}$ is an atom of $\mathscr{B}$. This is easily proved by constructing $\mathscr{B}$ isomorphic to the free product (Boolean product; [10], § 13) of $\mathscr{A}$ with a four-element Boolean algebra. The second notion we use is that of a perfect extension. $\mathscr{B}$ is called a *perfect extension* of $\mathscr{A}$ in case $\mathscr{A}$ is a subalgebra of $\mathscr{B}$, $\mathscr{B}$ is complete and atomic, and if an element a of $\mathscr{A}$ is the join in $\mathscr{B}$, of a subset $\mathscr{S}$ of $\mathscr{A}$, then a is the join of some finite subset of $\mathscr{S}$.

Every Boolean algebra $\mathscr{A}$ has a perfect extension $\mathscr{B}$. To prove this it suffices to construct $\mathscr{B}$ isomorphic to the field of all subsets of the Stone space of $\mathscr{A}$. The fact that every join, in $\mathscr{B}$, of elements of $\mathscr{A}$, is essentially finite follows easily from the compactness of the Stone space.

We let λ be the least ordinal of cardinality $\mathfrak{n}^+$, and define two λ termed sequences of Boolean algebras as follows:

$\mathscr{A}_0$ is the field of all subsets of a denumerable set,

$\mathscr{B}_0$ is a perfect extension of $\mathscr{A}_0$,

for $\alpha < \lambda$, $\mathscr{A}_{\alpha+1}$ is an extension of $\mathscr{B}_\alpha$ such that no atom of $\mathscr{B}_\alpha$ is an atom of $\mathscr{A}_{\alpha+1}$,

$\mathscr{B}_{\alpha+1}$ is a perfect extension of $\mathscr{A}_{\alpha+1}$,

for any limit ordinal $\gamma < \lambda$, $\mathscr{A}_\gamma = \mathscr{B}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{A}_\alpha$.

Note that for $\alpha < \beta < \lambda$ we have $\mathscr{A}_\alpha \subseteq \mathscr{B}_\alpha \subseteq \mathscr{A}_\beta$.

Finally, we let

$$\mathscr{C} = \bigcup_{\alpha<\lambda} \mathscr{A}_\alpha.$$

$\mathscr{C}$ is atomless and hence not in CSR. For if $a \in \mathscr{C}$, then choose $\alpha < \lambda$ such that $a \in \mathscr{A}_\alpha$. $a \in \beta_\alpha$, so a is not an atom of $\mathscr{A}_{\alpha+1}$. It follows that a is not an atom of $\mathscr{C}$. It remains to show that $\mathscr{C} \in K_\mathfrak{n}$. Suppose $a = \sum_{t \in T} a_t$ is an $\mathfrak{n}$-join which holds in $\mathscr{C}$. There is an ordinal $\beta < \lambda$ such that, for all $t \in T$, $a_t \in \mathscr{A}_\beta$, and $a \in \mathscr{A}$. Observe that $a = \sum_{t \in T} a_t$ holds in each of $\mathscr{A}_\beta$, $\mathscr{A}_{\beta+1}$ and $\mathscr{B}_{\beta+1}$. Since $\mathscr{B}_{\beta+1}$ is a perfect extension of $\mathscr{A}_{\beta+1}$, there is a finite set $F \subseteq T$ such that $a = \sum_{t \in F} a_t$ in $\mathscr{B}_{\beta+1}$. $a = \sum_{t \in F} a_t$ holds in $\mathscr{C}$ also. This proves that $\mathscr{C} \in C_\mathfrak{n}$, hence $\mathscr{C} \in K_\mathfrak{n}$.

The Boolean algebra $\mathscr{C}$ of Example 1 has large cardinality. We now turn to the problem of finding an element of $K_\mathfrak{n} - \mathrm{CSR}$ which has the smallest possible cardinality. This smallest cardinal is $\mathfrak{n}^+$. Let $\mathscr{A} \in K_\mathfrak{n}$. If $\mathscr{A}$ has cardinality at most $\mathfrak{n}$, then every nowhere dense subset, of the Stone space of $\mathscr{A}$, is actually $\mathfrak{n}$-nowhere dense. It follows that $\mathscr{A} \in \mathrm{CSR}$.

We need the notions of universal and homogeneous. Let $\mathscr{B}$ be a Boolean algebra of cardinality $\mathfrak{n}^+$. $\mathscr{B}$ is called *universal* if every Boolean algebra of cardinality at most $\mathfrak{n}^+$ is isomorphic to some subalgebra of $\mathscr{B}$. $\mathscr{B}$ is called *homogeneous* if for every subalgebra $\mathscr{A}$ of $\mathscr{B}$ whose cardinality is at most $\mathfrak{n}$, and every isomorphism h from $\mathscr{A}$ into $\mathscr{B}$, there is an automorphism of $\mathscr{B}$ which is an extension of h. These are special cases of general notions due to Jónsson ([2] and [3]). Jónsson proved that, under certain conditions on a class S of relational systems and under the assumption that $\mathfrak{n}^+ = 2^\mathfrak{n}$, there is a unique relational systems in S which is universal and homogeneous of cardinality $\mathfrak{n}^+$. The class of all Boolean algebras does satisfy Jónsson's conditions. The amalgamation property was proved by Dwinger and Yaqub [1] and the others are obvious. Jónsson's results do not tell us which Boolean algebra is universal and homogeneous. Keisler [4] identified the universal homogeneous Boolean algebra of cardinality $\aleph_1 = 2^{\aleph_0}$ as being the quotient algebra of the field of all subsets of $\aleph_0$ modulo its ideal of finite sets. This is the Boolean algebra which

Sierpiński showed to be in $K_{\aleph_0}$—CSR (see [6] and [7]). It turns out that the universal homogeneous Boolean algebra of cardinality $\mathfrak{n}^+$ is in $K_{\mathfrak{n}}$—CSR.

Example 2. Assume that $\mathfrak{n}^+ = 2^{\mathfrak{n}}$ and let $\mathscr{B}$ be universal homogeneous of cardinality $\mathfrak{n}^+$. $\mathscr{B}$ is atomless, for suppose b is an atom of $\mathscr{B}$. The four-element subalgebra of $\mathscr{B}$ which is generated by b has an obvious automorphism which cannot be extended to an automorphism of $\mathscr{B}$. This contradicts the homogeneous property of $\mathscr{B}$. We next show that $\mathscr{B} \in C_{\mathfrak{n}}$. Suppose that $\mathscr{B}$ has an $\mathfrak{n}$-join which is not essentially finite. There is a set $\{a_t: t \in T\}$ of elements of $\mathscr{B}$ such that T has cardinality at most $\mathfrak{n}$, $\sum_{t \in T} a_t = 1$ (the unit element of $\mathscr{B}$), and the $\mathfrak{n}$-join $1 = \sum_{t \in T} a_t$ is not essentially finite. Let $\mathscr{A}$ be the subalgebra of $\mathscr{B}$ which is generated by $\{a_t: t \in T\}$. $\mathscr{A}$ has cardinality at most $\mathfrak{n}$, and for every finite set $F \subseteq T$, $1 \neq \sum_{t \in F} a_t$ in $\mathscr{A}$ also. By Zorn's Lemma there is a maximal proper ideal J of $\mathscr{A}$ satisfying $\{a_t: t \in T\} \subseteq J$. Let $\mathscr{C}$ be the direct sum of $\mathscr{A}$ with the two-element Boolean algebra. The elements of $\mathscr{C}$ are ordered pairs (a, i) with $a \in \mathscr{A}$ and $i \in \{0, 1\}$, and the operations of $\mathscr{C}$ are componentwise. We define a function h from $\mathscr{A}$ into $\mathscr{C}$ by

$$h(a) = \begin{cases} (a, 0) & \text{if } a \in J, \\ (a, 1) & \text{if } a \notin J. \end{cases}$$

h is easily seen to be an isomorphism from $\mathscr{A}$ into $\mathscr{C}$. Note that $(1, 0)$ is an upper bound, in $\mathscr{C}$, of $\{h(a_t): t \in T\}$. Since $\mathscr{B}$ is universal, there is an isomorphism g from $\mathscr{C}$ into $\mathscr{B}$. Since $\mathscr{B}$ is homogeneous, there is an automorphism f of $\mathscr{B}$ which extends $g \circ h$. Now $1 = \sum_{t \in T} a_t$ in $\mathscr{B}$ implies that

$$1 = \sum_{t \in T} f(a_t) = \sum_{t \in T} g \circ h(a_t).$$

But for all $t \in T$ we have $h(a_t) \leqslant (1, 0)$, so that $g \circ h(a_t) \leqslant g(1, 0) \neq 1$. This contradiction proves that $\mathscr{B} \in C_{\mathfrak{n}}$. Hence $\mathscr{B} \in K_{\mathfrak{n}}$—CSR.

We next consider the question of whether a Boolean algebra in $K_{\mathfrak{n}}$—CSR can be $\mathfrak{n}$-complete. We show that it cannot by proving the following

THEOREM. *Let $\mathscr{A} \in K_{\mathfrak{n}}$ and suppose $\mathscr{A}$ is $\aleph_0$-complete. Then $\mathscr{A}$ is atomic.*

Proof. Since $\mathscr{A} \in K_{\mathfrak{n}}$, $\mathscr{A}$ is weakly $(\mathfrak{m}, \mathfrak{n})$-distributive for every cardinal number $\mathfrak{m}$. We assume that there is an element $a \neq 0$ of $\mathscr{A}$ for which no atom b of $\mathscr{A}$ satisfies $b \leqslant a$. We define M to be the set of all subsets D of $\mathscr{A}$ satisfying the following conditions:

(1) for each $d \in D$, $0 \neq d \leqslant a$;
(2) distinct elements of D are disjoint;
(3) $a = \sum_{d \in D} d$;
(4) D has cardinality $\aleph_0$.

M is not empty. For the principle ideal of $\mathscr{A}$, which is generated by a, forms an infinite Boolean algebra $\mathscr{B}$ with a as its unit element. A well-known theorem states that every infinite Boolean algebra contains an infinite set of pairwise disjoint elements. (A proof of this can be found in [5].) An element of M is now easily constructed using the $\aleph_0$-completeness of $\mathscr{A}$. We choose suitable index sets T and S, with S having cardinality $\aleph_0$, and for each $D \in M$ there is a unique $t \in T$ such that $D = \{d_{t,s} : s \in S\}$. We have

$$a = \prod_{t \in T} \sum_{s \in S} d_{t,s}.$$

Now consider any $\Phi \in (\operatorname{fin} S)^T$. We claim that

$$\prod_{t \in T} \sum_{s \in \Phi(t)} d_{t,s} = 0.$$

For if $x \neq 0$ and $x \leqslant \sum_{s \in \Phi(t)} d_{t,s}$ for all $t \in T$, then x is an atomless element of $\mathscr{A}$, as was the a above. x is then the join of a denumerable set G of pairwise disjoint, non-zero elements. Letting $y = a \cdot (-x)$ (the meet of a with the complement of x), we see that $G \cup \{y\} \in M$. Let t' be such that

$$G \cup \{y\} = \{d_{t',s} : s \in S\}.$$

It follows from the finiteness of $\Phi(t')$ that

$$x > \sum_{s \in \Phi(t')} d_{t',s}.$$

This contradiction proves

$$\prod_{t \in T} \sum_{s \in \Phi(t)} d_{t,s} = 0 \qquad \text{for every } \Phi \in (\operatorname{fin} S)^T.$$

Hence

$$\sum_{\Phi \in (\operatorname{fin} S)^T} \prod_{t \in T} \sum_{s \in \Phi(t)} d_{t,s} = 0,$$

and this contradiction to the weak $(\mathfrak{m}, \mathfrak{n})$-distributivity of $\mathscr{A}$ proves that $\mathscr{A}$ is atomic.

Concluding remarks. If $\mathfrak{m} > \mathfrak{n}$, then $K_{\mathfrak{m}}$ is properly contained in $K_{\mathfrak{n}}$, because the Boolean algebra $\mathscr{C}$, of Example 2, has cardinality too small to be in $K_{\mathfrak{m}}$. Evidently CSR is the intersection of all the $K_{\mathfrak{n}}$.

$\mathfrak{n}$SR is also properly contained in $K_\mathfrak{n}$. For let $\mathscr{A}$ be any atomless $\mathfrak{n}$-field of sets. By the theorem, $\mathscr{A}$ is not in $K_\mathfrak{n}$. The class $C_\mathfrak{n}$ is properly contained in $K_\mathfrak{n}$, for let $\mathscr{A} \in C_\mathfrak{n}$ and let $\mathscr{B}$ be the direct sum of $\mathscr{A}$ with the field of all subsets of a set of cardinality $\mathfrak{n}$. It is easily seen that $\mathscr{B} \in K_\mathfrak{n}$ and $\mathscr{B} \notin C_\mathfrak{n}$. If, in addition, $\mathscr{A} \notin$ CSR, then $\mathscr{B} \notin$ CSR also.

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