COLLOQUIUM MATHEMATICUM

VOL. XXIII 1971 FASC. 2

ON NON LOCALLY p-CONVEX SPACES

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There are known examples of complete linear metric spaces (or F-spaces) none of whose infinite-dimensional subspaces are locally convex. In fact, it has been shown in [2] and [5] that for each p, $0 < p < 1$, each infinite-dimensional subspace of l_p contains a subspace isomorphic to l_p. Hence no infinite-dimensional subspace of l_p can be locally convex. In addition, it has been shown in [1] that if an F-space with a basis has the property that every infinite-dimensional subspace contains a subspace isomorphic to s (the space of all real sequences with the topology of point-wise convergence), then the space itself must be isomorphic to s. In view of these facts it seems natural to ask whether there is an example of an F-space (with a basis) which is not locally convex but which has the property that every infinite-dimensional subspace contains an infinite-dimensional subspace which is locally convex. The answer to this question is affirmative, and such a space will be constructed below.

We first recall that for any sequence $\{p_n\}$, $0 < p_n \leqslant 1$, the space $l_{\{p_n\}}$ is the set of all sequences, $x = (x_i)$, such that $\|x\| = \sum |x_i|^{p_i} < \infty$. We also recall that a set S is *p-convex* if and only if $\sum_1^n |a_i|^p \leqslant 1$ and $s_i \in S$ implies $\sum_1^n a_i s_i \in S$, and that a linear topological space is *locally p-convex* if and only if it contains a local base at the origin which is made up of p-convex sets. (It is clear that l_p is locally p-convex.)

THEOREM. *For each p, $0 < p \leqslant 1$, there is a complete linear metric space with a basis which is not locally p-convex but is such that each infinite-dimensional subspace contains an infinite-dimensional subspace whose closure is isomorphic to l_p.*

Proof. For $0 < p \leqslant 1$, choose an increasing sequence $\{q_n\}$ such that $\lim q_n = p$ and

$$\sup_{0 \leqslant t \leqslant 1} |t^p - t^{q_n}| < \frac{1}{2^n}.$$

Let $\{k_n\}$ be a sequence of positive integers such that $k_n^{1-q_n/p} > n$; and, for each positive integer n, let $p_n = q_m$ if

$$\sum_{j=0}^{m-1} k_j < n \leqslant \sum_{j=0}^{m} k_j, \quad \text{where } k_0 = 0.$$

It is easy to see that $l_{\{p_n\}}$ forms a non locally p-convex space where the unit vectors form a basis. (An alternate construction is given in [3].) Let X be an infinite-dimensional subspace of $l_{\{p_n\}}$. Theorem B.II.5.7 and Theorem B.II.5.8 of [4] imply that there is a sequence $\{b_n\}$ such that $b_n = (b_1^n, b_2^n, \ldots)$, $b_n \in X$, $\|b_n\| = 1$, and a sequence $\{r_n\}$ of indices such that

$$(*) \qquad \|b_n - c_n\| < \frac{1}{2^{n+1}},$$

where $c_n = (0, \ldots, b_{r_n}^n, \ldots, b_{r_{n+1}-1}^n, 0, \ldots)$. Since $\{c_n\}$ is a block basis and the norm in $l_{\{p_n\}}$ is increasing, $(*)$ implies that $\{b_n\}$ is a basic sequence equivalent to $\{c_n\}$.

To complete the proof of the theorem, we show that we can choose $\{r_n\}$ such that $\{c_n\}$ is also equivalent to the unit vector basis of l_p. First, the convergence of $\sum \lambda_n c_n$ implies the convergence of $\sum |\lambda_n|^p$ because $\|\lambda_n c_n\| \geqslant |\lambda_n|^p \|c_n\|$ when $|\lambda_n| \leqslant 1$. Finally, the opposite holds because $\sum |\lambda_n|^p < \infty$ and $|\lambda_n| \leqslant 1$ implies

$$\|\lambda_n c_n\| = \sum_{j=r_n}^{r(n+1)-1} |\lambda_n|^{p_j} |b_j^n|^{p_j} \leqslant \sum_{j=r_n}^{r(n+1)-1} \left(|\lambda_n|^p - \frac{1}{2^n}\right) |b_j^n|^{p_j} \leqslant |\lambda_n|^p - \frac{1}{2^n}.$$

We are indebted to S. Rolewicz for noting that our original proof for $p = 1$ could easily be extended to all p, $0 < p \leqslant 1$.

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Reçu par la Rédaction le 26. 1. 1970;
en version modifiée le 27. 4. 1970