COLLOQUIUM MATHEMATICUM

XVI DÉDIÉ À M. FRANCISZEK LEJA 1967

ON THE USE OF THE GRUNSKY-NEHARI INEQUALITY FOR ESTIMATING THE FOURTH COEFFICIENT OF BOUNDED UNIVALENT FUNCTIONS

BY

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1. In a former paper [3] Grunsky-Nehari inequality [1] was used to estimate the fourth coefficient a_4 of functions

$$F(z) = z+a_2z^2+\ldots,\ |z|<1, \tag{1}$$

which are schlicht and bounded:

$$|F(z)| \leqslant b_1^{-1} \quad (0<b_1\leqslant 1).$$

The inequality in question reads (cf. [3], (4)).

$$\begin{aligned}\operatorname{Re}\{a_4-2a_2a_3+\tfrac{13}{12}a_2^3+x_1^2a_2+2x_1(a_3-\tfrac{3}{4}a_2^2)\} \\ \leqslant \tfrac{2}{3}(1-b_1^3)-\tfrac{1}{2}b_1|a_2|^2+2|x_1|^2(1-b_1)-2b_1\operatorname{Re}\{x_1\bar{a}_2\}+ \\ +|x_2|^2(1-b_1^2)-\operatorname{Re}\{x_2^2(a_3-a_2^2)\}.\end{aligned} \tag{2}$$

Here x_1 and x_2 are arbitrary complex parameters. The aim of the present paper is to utilize this inequality in the best possible manner near the point $b_1=0$.

First we notice that a well-known coefficient inequality gives (cf. [4], (38))

$$|x_2^2(a_3-a_2^2)| \leqslant |x_2^2(1-b_1^2)| = |x_2|^2(1-b_1^2),$$
$$|x_2|^2(1-b_1^2)-\operatorname{Re}\{x_2^2(a_3-a_2^2)\} \geqslant 0.$$

This shows that in estimating $\operatorname{Re}\{a_4\}$, the best possible choice is always $x_2=0$. In former applications one took $x_1=l=$ real.

The best possible choice for x_1 is obviously obtained by generalizing:

$$x_1 = l+im; \quad l, m \text{ are real.} \tag{3}$$

From (2) we deduce

$$\text{(4)} \qquad \operatorname{Re}\{a_4\} \leqslant \frac{2}{3}(1-b_1^3)-\frac{1}{2}b_1|a_2|^2+\frac{5}{12}\operatorname{Re}\{a_2^3\}+$$
$$+2\operatorname{Re}\{a_2\lambda\}+2|x_1|^2(1-b_1)-\operatorname{Re}\{x_1^2a_2\}-2\operatorname{Re}\{x_1\lambda\}-$$
$$-2b_1\operatorname{Re}\{x_1\bar{a}_2\}, \qquad \lambda = a_3-\frac{3}{4}a_2^2.$$

The part of this upper bound depending on x_1 and a_3 is denoted by

$$\text{(5)} \qquad H = 2\operatorname{Re}\{a_2\lambda\}+2|x_1|^2(1-b_1)-\operatorname{Re}\{x_1^2a_2\}-$$
$$-2\operatorname{Re}\{x_1\lambda\}-2b_1\operatorname{Re}\{x_1\bar{a}_2\}.$$

Write further

$$\text{(6)} \qquad a_2 = u+iv, \qquad \lambda = U+iV$$

and thus get

$$\text{(7)} \qquad H = 2Uu-2Vv-2(U+b_1u)l+e_2l^2+2(lv+V-b_1v)m+e_1m^2,$$
$$e_1 = 2(1-b_1)+u, \qquad e_2 = 2(1-b_1)-u.$$

Now specialize m, which is an arbitrary real number so as to minimize H for every fixed l. We have $e_1 > 0$ except the case of right radial slit. In all other cases the best possible choice of m is to take

$$\text{(8)} \qquad m = -\frac{(l-b_1)v+V}{e_1}.$$

This gives for H the equation

$$H = 2Uu-2Vv-2(U+b_1u)l+e_2l^2-\frac{(V-b_1v+lv)^2}{e_1}.$$

By rearranging terms we get

$$\text{(9)} \qquad H = 2Uu-2Vv-\frac{(V-b_1v)^2}{e_1}-2\left[U+b_1u+\frac{(V-b_1v)v}{e_1}\right]l+$$
$$+\frac{4(1-b_1)^2-u^2-v^2}{e_1}l^2.$$

Here the coefficient of l^2 is positive except the cases of radial slit. In all other cases we can again minimize H as a function of l. The result is

$$\text{(10)} \qquad H = 2Uu-2Vv-\frac{(V-b_1v)^2}{e_1}-$$
$$-\frac{e_1}{4(1-b_1)^2-u^2-v^2}\left[U+b_1u+\frac{(V-b_1v)v}{e_1}\right]^2.$$

Would we now follow the former lines, we had to maximize H as a function of U. The upper bound thus obtained for $\mathrm{Re}\{a_4\}$ would then further be maximized by the aid of an inequality given by the area principle (cf. [3], (21)). We will first try to proceed in a different way.

As in [3], introduce the combination

$$\mu = \lambda + b_2. \tag{11}$$

Write

$$\mathrm{Re}\{\mu\} = \xi, \quad \mathrm{Im}\{\mu\} = \eta. \tag{12}$$

Hence we have $U = \xi - b_1 u$, $V = \eta - b_1 v$. Substituting this in (10) we express H in terms of ξ and η:

$$H = 2u(\xi - b_1 u) - 2v(\eta - b_1 v) - \frac{(\eta - 2b_1 v)^2}{e_1} -$$
$$- \frac{e_1}{4(1-b_1^2)-u^2-v^2}\left[\xi + \frac{(\eta-2b_1 v)v}{e_1}\right]^2 = -\frac{e_1}{\Delta}\xi^2 - \frac{e_2}{\Delta}\eta^2 - 2\frac{v}{\Delta}\xi\eta +$$
$$+2\left(u + \frac{2b_1 v^2}{\Delta}\right)\xi + 2\left(-v + \frac{2b_1 e_2 v}{\Delta}\right)\eta + 2b_1\left(-u^2 + v^2 - \frac{2b_1 e_2 v^2}{\Delta}\right),$$
$$\Delta = 4(1-b_1)^2 - u^2 - v^2.$$

Combine this result with (5):

$$\mathrm{Re}\{a_4\} \leqslant \frac{2}{3}(1-b_1^3) - \frac{1}{2}b_1(u^2+v^2) + \frac{5}{12}(u^3 - 3uv^2) - 2b_1 u^2 - \frac{e_1}{\Delta}\xi^2 -$$
$$-\frac{e_2}{\Delta}\eta^2 - 2\frac{v}{\Delta}\xi\eta + 2\left(u + \frac{2b_1 v^2}{\Delta}\right)\xi + 2\left(-v + \frac{2b_1 e_2 v}{\Delta}\right)\eta +$$
$$+2b_1\left(v^2 - \frac{2b_1 e_2 v^2}{\Delta}\right).$$

Finally, take new variables as follows:

$$\begin{aligned} x &= 2(1-b_1) - u = e_2, \quad 0 < x \leqslant 2(1-b_1), \\ y &= v. \end{aligned} \tag{13}$$

Thus we have proved

THEOREM 1. *Except the cases of radial slit, we have*

$$\mathrm{Re}\{a_4\} \leqslant 4 - 20b_1 + 30b_1^2 - 14b_1^3 + K, \tag{14}$$

where

$$K = A\xi^2+B\eta^2+2C\xi\eta+2D\xi+2E\eta+F,$$

$$A = -\frac{4(1-b_1)-x}{\Delta}, \quad B = -\frac{x}{\Delta}, \quad C = -\frac{y}{\Delta},$$

$$D = 2(1-b_1)-x+\frac{2b_1y^2}{\Delta}, \quad E = -y+\frac{2b_1xy}{\Delta},$$

$$F = -5(1-b_1)(1-3b_1)x+\frac{5}{2}(1-2b_1)x^2-\frac{5}{12}x^3+$$

$$+\left[-\frac{5}{2}+4b_1+\left(\frac{5}{4}-\frac{4b_1^2}{\Delta}\right)x\right]y^2,$$

$$\Delta = 4(1-b_1)x-x^2-y^2.$$

2. Next we will try to utilize the upper bound of (14) in its strongest form. In order to maximize K keep first x and y fixed. For them we have

$$[x-2(1-b_1)]^2+y^2 \leqslant [2(1-b_1)]^2,$$

$$x^2+y^2-4(1-b_1)x \leqslant 0.$$

In K, the change of the sign of y can be compensated by the corresponding change of η. Thus we may take

$$(15) \qquad 0 \leqslant y < \sqrt{4(1-b_1)x-x^2} = y_0, \quad 0 < x \leqslant 2(1-b_1).$$

Notice that if $y = y_0$, then $\Delta = 0$. This belongs thus to the radial slit case which here is excluded. For x and y fixed, consider K as a function of ξ and η. Because of condition (27) of [3] we have for these arguments

$$(16) \qquad \xi^2+\eta^2 \leqslant \frac{4x-x^2-y^2}{3} = R^2.$$

Our task will thus be to maximize

$$(17) \qquad K(\xi,\eta) = A\xi^2+B\eta^2+2C\xi\eta+2D\xi+2E\eta+F$$

in the closed circle $\xi^2+\eta^2 \leqslant R^2$. Consider the free maximum point $P_0(\xi_0,\eta_0)$ for which

$$\frac{1}{2}\frac{\partial K}{\partial \xi} = A\xi+C\eta+D = 0, \quad \frac{1}{2}\frac{\partial K}{\partial \eta} = C\xi+B\eta+E = 0;$$

thus

$$\xi_0 = \Delta(CE-BD) = 2(1-b_1)x-x^2+y^2,$$

$$\eta_0 = \Delta(CD-AE) = 2(-3+4b_1+x)y.$$

We have

$$(18)\qquad K(\xi_0,\eta_0) = D\xi_0+E\eta_0+F$$
$$= (1-b_1)(-1+11b_1)x-\left(\frac{3}{2}+b_1\right)x^2+\frac{7}{12}x^3+\left(\frac{11}{2}-8b_1-\frac{7}{4}x\right)y^2.$$

We have to study the sign of this function.

The coefficient $\frac{11}{2}-8b_1-\frac{7}{4}x$ is positive, when $b_1<\frac{4}{9}$. Thus we see that the number $K(\xi_0,\eta_0)$ changes its sign from negative to positive when y^2 grows over the value

$$(19)\qquad \tilde{y}^2 = \frac{(1-b_1)(1-11b_1)x+(\frac{3}{2}+b_1)x^2+\frac{7}{12}x^3}{\frac{11}{2}-8b_1-\frac{7}{4}x} > 0.$$

In order to determine the location of $\tilde{y}^2$ ask when $\tilde{y}^2 \leqslant y_0^2$. It is seen that this condition is fulfilled for

$$0 < x \leqslant 6\left(1-\frac{\sqrt{2}}{2}\right)(1-b_1).$$

For those values of x the critical value $y=\tilde{y}$ lies inside the domain (15). This means that $K(\xi_0,\eta_0)$ has positive values for $y>\tilde{y}$. Ask now what is the location of the point $P_0(\xi_0,\eta_0)$ with respect to the circle $\xi^2+\eta^2\leqslant R^2$. Take especially the critical curve $y=\tilde{y}(x)$ and get in this case

$$\xi_0 = 2(1-b_1)x-x^2+\frac{(1-b_1)(1-11b_1)x+(\frac{3}{2}+b_1)x^2+\frac{7}{12}x^3}{\frac{11}{2}-8b_1-\frac{7}{4}x},$$

$$\eta_0 = 2(-3+4b_1+x)\sqrt{\frac{(1-b_1)(1-11b_1)x+(\frac{3}{2}+b_1)x^2+\frac{7}{12}x^3}{\frac{11}{2}-8b_1-\frac{7}{4}x}},$$

$$3R^2 = 4x-x^2-\frac{(1-b_1)(1-11b_1)x+(\frac{3}{2}+b_1)x^2+\frac{7}{12}x^3}{\frac{11}{2}-8b_1-\frac{7}{4}x.}.$$

Now, taking $b_1=\frac{1}{11}$ we get

$$\xi_0^2+\eta_0^2 = kx^2+\ldots\ (k>0),\qquad R^2=\tfrac{4}{3}x+\ldots$$

Thus, for x near 0 the point P_0 lies inside the circle $\xi^2+\eta^2\leqslant R^2$. This means that, together with $K(\xi_0,\eta_0)$, the function $K(\xi,\eta)$ gets positive values in the domain (15) provided b_1 is near enough to the point $\frac{1}{11}$. Thus, Grunsky-Nehari inequality (2) cannot be used to extend the result obtained in the symmetric class (cf. [2]) to the general class.

3. Estimation (14) is better that that utilized in [3]. Hence, there is still left the possibility of trying to use (14) in a way similar to that of [3]. To this purpose maximize K on the right-hand side of (14) as a function of ξ and consider the result as a function of η:

$$K = K_{\max} = \left(B-\frac{C^2}{A}\right)\eta^2+2\left(E-\frac{CD}{A}\right)\eta+F-\frac{D^2}{A}.$$

By substituting here the expressions of the coefficients $A, B, \ldots$ and by rearranging the terms we get

$$\begin{aligned}(20)\quad & \mathrm{Re}\{a_4\}-(4-20b_1+30b_1^2-14b_1^3)\\ & \leqslant (1-b_1)(-1+11b_1)x-\tfrac{1}{2}(3+2b_1)x^2+\tfrac{7}{12}x^3+\\ & +\frac{-56+168b_1-128b_1^2+(46-68b_1)x-9x^2}{4[4(1-b_1)-x]}y^2-\\ & -4\frac{3-4b_1-x}{4(1-b_1)-x}\eta y-\frac{1}{4(1-b_1)-x}\eta^2.\end{aligned}$$

Observe that here

$$(21)\qquad \frac{3-4b_1-x}{4(1-b_1)-x}>0 \quad \text{for} \quad b_1<\frac{1}{2}.$$

By the aid of a positive parameter a estimate

$$-2y\eta = -a\left(y+\frac{1}{2}\eta\right)^2+ay^2+\frac{1}{a}\eta^2 < ay^2+\frac{1}{a}\eta^2 \qquad (a>0).$$

Hence

$$\begin{aligned}& -4\frac{3-b_1-x}{4(1-b_1)-x}\eta y-\frac{1}{4(1-b_1)-x}\eta^2\\ & < 2\frac{3-4b_1-x}{4(1-b_1)-x}\left(ay^2+\frac{1}{a}\eta^2\right)-\frac{1}{4(1-b_1)-x}\eta^2\\ & = \frac{6-8b_1-2x}{4(1-b_1)-x}ay^2+\frac{(6-8b_1-2x)\frac{1}{a}-1}{4(1-b_1)-x}\eta^2;\end{aligned}$$

$$\begin{aligned}(22)\quad & \mathrm{Re}\{a_4\}-(4-20b_1+30b_1^2-14b_1^3)-\\ & -\left[(1-b_1)(-1+11b_1)x-\frac{1}{2}(3+2b_1)x^2+\frac{7}{12}x^3\right]\\ & < \frac{T+(6-8b_1-2x)a}{4(1-b_1)-x}y^2+\frac{(6-8b_1-2x)\frac{1}{a}-1}{4(1-b_1)-x}\eta^2;\\ & T = -14+42b_1-32b_1^2+\frac{1}{2}(23-34b_1)x-\frac{9}{4}x^2.\end{aligned}$$

According to (16) we have

$$\eta^2 \leqslant \frac{4x-x^2}{3} - \frac{1}{3}y^2. \tag{23}$$

This estimation can be utilized on the right-hand side of (22) provided that

$$(6-8b_1-2x)\frac{1}{a} - 1 > 0$$

or

$$0 < a < 6-8b_1-2x. \tag{24}$$

Suppose that this is true and estimate the right-hand side of (22) by the aid of (23):

$$\begin{aligned}(25)\quad & \operatorname{Re}\{a_4\} - (4-20b_1+30b_1^2-14b_1^3) - \\ & -\left[(1-b_1)(-1+11b_1)x - \frac{1}{2}(3+2b_1)x^2 + \frac{7}{12}x^3\right] \\ & < \frac{T+\frac{1}{3}+(6-8b_1-2x)(a-1/3a)}{4(1-b_1)-x}\,y^2 + \frac{(6-8b_1-2x)\frac{1}{a}-1}{4(1-b_1)-x}\cdot\frac{4x-x^2}{3}.\end{aligned}$$

Similarly to the procedure of [3], determine now a so that the coefficient of y^2 disappears:

$$\frac{1}{3a} - a + \alpha = 0, \qquad \alpha = -\frac{T+\frac{1}{3}}{6-8b_1-2x};$$

$$\begin{aligned}(26)\quad & a = \frac{\alpha+\sqrt{\alpha^2+\frac{4}{3}}}{2}, \\ & \alpha = \frac{\frac{41}{3}-42b_1+32b_1^2-\frac{1}{2}(23-34b_1)x+\frac{9}{4}x^2}{6-8b_1-2x}.\end{aligned}$$

Now we have to check the validity of condition (23), i.e. we have to verify (24):

$$\frac{\alpha+\sqrt{\alpha^2+\frac{4}{3}}}{2} < 6-8b_1-2x,$$

or

$$\sqrt{\alpha^2+\frac{4}{3}} < 2(6-8b_1-2x)-\alpha. \tag{27}$$

Note first that according to (21) we have $6-8b_1-2x > 0$. Moreover, we can show that the right-hand side of (27) is positive for $b_1 < \frac{1}{2}$. Thus, by squaring we find that (24) is true if

$$22-54b_1+32b_1^2+\left(-\frac{25}{2}+15b_1\right)x+\frac{7}{4}x^2 > 0\,.$$

It is seen that this condition is fulfilled at least for $b_1 < \frac{1}{3}$. Thus we have proved that by choosing a according to (26) the upper bound (25) is attainable. Write the result in the following form:

THEOREM 2. *Except the radial slit cases, we have for $b_1 < \frac{1}{3}$ the estimation*

$$\mathrm{Re}\{a_4\}-4-20b_1+30b_1^2-14b_1^3) < \varrho(x)+\delta(x)\,; \tag{28}$$

$$\varrho(x) = (1-b_1)(-1+11b_1)x-\frac{1}{2}(3+2b_1)x^2+\frac{7}{12}x^3\,;$$

$$\delta(x) = \frac{4x-x^2}{4(1-b_1)-x}\left[(3-4b_1-x)\left(\sqrt{a^2+\frac{4}{3}}-a\right)-\frac{1}{3}\right],$$

$$a = \frac{\frac{41}{3}-42b_1+32b_1^2-\frac{1}{2}(23-34b_1)x+\frac{9}{4}x^2}{6-8b_1-2x}\,.$$

Numerical evaluation of the function $\varrho(x)+\delta(x)$ for $0 < x \leqslant 2(1-b_1)$ shows that the inequality

$$|a_4| \leqslant 4-20b_1+30b_1^2-14b_1^3$$

can be extended to the interval

$$0 \leqslant b_1 \leqslant \frac{1}{25}\,.$$

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ON A THEOREM OF KUBO CONCERNING FUNCTIONS REGULAR IN AN ANNULUS

BY

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1. Introduction. As pointed out by Hayman [1], the notion of the transfinite diameter of a compact set in the plane and its properties such as monotonic behaviour under expansion, resp. symmetrization of the set considered, give rise to a method very useful in tackling certain extremal problems connected with functions meromorphic in the unit disk K_1. According to Hayman, the transfinite diameter $d(E_\varphi)$ of the set E_φ of values omitted by

$$\varphi(z) = z^{-1}+a_0+a_1z+\ldots$$

meromorphic in K_1 does not surpass 1 and the equality $d(E_\varphi) = 1$ holds if and only if φ is univalent.

A few years ago T. Kubo obtained an analogous theorem for functions regular in an annulus [5] which yields Hayman's results in the limiting case of an annulus shrinking to a punctured disk. The result of Kubo is related to a sort of subordination principle for multiply connected domains due to M. Schiffer (cf. [3], p. 92). The theorem of Kubo has been recently rediscovered by Mityuk [7], no proof has been, however, published.

The hyperbolic transfinite diameter $\tau(E)$ of a compact subset E of the unit disk introduced by Tsuji [8] may be defined as a generalized transfinite diameter in the sense of F. Leja (cf. [6], p. 258), with the generating function $\omega(z, \zeta) = |z-\zeta|\,|1-z\zeta|^{-1}$. The hyperbolic transfinite diameter $\tau(E)$ has similar properties as the transfinite diameter. It is invariant under hyperbolic motion, and decreases under circular and Steiner symmetrization and shrinking of the set. Besides, we have $\tau(\overline{K}_r) = r$ where $K_r = \{z: |z| < r\}$ and $0 < r < 1$. For other definitions and propr erties, cf. [9]. Let $R(q)$ be the class of functions regular in the annulus $B_q = \{z: q < |z| < 1\}$ which satisfy $|f(z)| < 1$ in B_q and map the unit circumference $|z| = 1$ in a continuous manner onto the unit circumference $|w| = 1$. It is easy to see that the set of values not taken by f consists of the closed, unbounded set $E_0 = \{w: |w| \geqslant 1\}$ and of a compact set E_f contained inside the unit disk K_1. Now, according to Kubo, we have the following

THEOREM. *If $f \in R(q)$, then $\tau(E_f) \leqslant q$. Equality holds only for functions univalent in B_q.*

Kubo's proof is based on the properties of the so called hyperbolic conductor potential of E_f. The aim of this paper is to give an alternative proof of Kubo's theorem by the method of extremal length.

2. Hyperbolic capacity as a module. We call a *condenser* a pair $\{E_0, E_1\}$ of two closed and disjoint sets (not necessarily connected) such that $\infty \in E_0$ and the union $E_0 \cup E_1$ is the complementary set of a planar domain Ω which is called the *field* of $\{E_0, E_1\}$. If Ω is regular with respect to the Dirichlet problem, we call the condenser $\{E_0, E_1\}$ *regular*. In what follows we assume $E_0 = \{z: |z| \geqslant 1\}$. Suppose $\{E_0, E_1\}$ is a regular condenser. It is well known that the Dirichlet integral

$$I(w) = \iint_{\Omega} (w_x^2 + w_y^2)\, dx dy,$$

where $w = w(z, E_1)$ denotes the harmonic measure of E_1 with respect to Ω, is finite [2]. We call $I(w)$ the *capacity* of the regular condenser $\{E_0, E_1\}$ and denote it by $|\{E_0, E_1\}|$. If E_1 is an arbitrary compact subset of K_1, the capacity $|\{E_0, E_1\}|$ is defined as the limit $\lim_{n\to\infty} |\{E_0, F_n\}|$, where $\{E_0, F_n\}$ are regular and $\{F_n\}$ is a decreasing sequence of compact sets with

$$\bigcap_{n=1}^{\infty} F_n = E_1.$$

It is well known that $|\{E_0, E_1\}|$ can be expressed by means of extremal length or their reciprocal (module) of a certain family of curves. Let $\{\gamma\}$ be the family of all locally rectifiable curves γ starting at points of $C = \{z: |z| = 1\}$, contained in Ω and tending in one sense to the set E_1. Consider the family P of all non-negative Borel measurable functions defined in Ω and such that

$$L(\varrho) = \inf_{\{\gamma\}} \int_{\gamma} \varrho(z)\, |dz|, \qquad A(\varrho, \Omega) = \iint_{\Omega} \varrho^2(z)\, dx dy$$

are not simultaneously 0 or ∞. Then we have

$$|\{E_0, E_1\}| = \inf_{\varrho \in P} \frac{A(\varrho, \Omega)}{L^2(\varrho)} = M\{\gamma\}. \tag{2.1}$$

If $|\{E_0, E_1\}| > 0$, then there exists the extremal metric $\tilde{\varrho}$ for which the g.l.b. in (2.1) is attained and $\tilde{\varrho}(z) = |w_x - i w_y|$, where $w(z)$ is the harmonic measure of E_1 with respect to the domain Ω which is equal to $\lim_n w(z, F_n)$. It is also well known, cf. [9], that if $|\{E_0, E_1\}| > 0$, then

$$\tau(E) = \exp(-2\pi|\{E_0, E_1\}|^{-1}). \tag{2.2}$$

Comparing this with (2.1) we see

$$\tau(E) = \exp(-2\pi M^{-1}\{\gamma\}). \tag{2.3}$$

This equation could be also used as a definition of the *hyperbolic transfinite diameter*. Instead of $\tau(E)$ we can also consider the expression

$$\gamma(E) = -\log\tau(E) = 2\pi M^{-1}\{\gamma\} \tag{2.4}$$

and call it *hyperbolic Robin's constant* in analogy to the logarithmic Robin's constant $C(E) = -\log d(E)$.

3. Proof of Kubo's theorem. First of all we may assume that $\tau(E_f) > 0$, or $|\{E_0, E_f\}| > 0$, since otherwise the theorem is trivial. Note that this may happen; E_f may be even empty (e.g. for the case of a conformal mapping of B_q onto a two-sheeted unit disk). In view of the conformal invariance of the Dirichlet integral and the relation

$$|\{E_0, \bar{K}_q\}|^{-1} = -\frac{\log q}{2\pi},$$

the equality $\tau(E_f) = q$ holds evidently for univalent f. Suppose now that $f(z)$ is not univalent in B_q. Then $f(z)$ is not univalent in a smaller annulus B_r, $q < r < 1$. Let $E_f(r)$ be the set of values w, $|w| < 1$, not taken by f in B_r. We construct à metric ϱ in $K_1 - E_f(r) = \Omega_r$ (depending on r) such that after removing a set of zero measure from Ω_r all the curves γ joining C to $E_f(r)$ and situated in the remaining set will have the length $\geqslant 1$ whereas the difference $|\{E_0, \bar{K}_r\}| - M\{\gamma\} \geqslant |\{E_0, \bar{K}_r\}| - A(\varrho, \Omega_r)$ will be estimated from below by a positive term, not decreasing if we replace r by a smaller number. This will imply $|\{E_0, \bar{K}_q\}| > M\{\gamma\} = |\{E_0, E_f\}|$, i.e. $\tau(\bar{K}_q) = q > \tau(E_f)$.

Consider now $f(z)$ in B_r. The curve Γ_r: $w = f(re^{i\theta})$, $0 \leqslant \theta \leqslant 2\pi$, is an analytic curve having no points in common with $|w| = 1$ which is a consequence of the domain invariance and the inequality $|f(z)| < 1$, $z \in B_q$. We now show that after removing a finite number of analytic arcs from Ω_r we obtain a subdivision of Ω_r into disjoint, simply connected domains which are mapped under $w = f(z)$ 1-1 onto a part of B_r. The complementary set of Γ_r with respect to K_1 is a union of a finite number of domains such that the index $n(a) = n(\Gamma_1 - \Gamma_r, a)$ of the cycle $\Gamma_1 - \Gamma_r$ has a constant value in each of these domains. The closure of $\{a: n(a) = 0\}$ is equal to $E_f(r)$. Besides, $\{a: n(a) \geqslant 2\}$ is not empty since $f(z)$ is not univalent in K_r. Only one component of $K_1 - \Gamma_r$ has the unit circumference as a part of boundary. We now slit the only doubly connected domain as well as all the remaining domains which are simply connected so as to obtain simply connected domains without points $w = f(z)$ for which $f'(z) = 0$. In view of the monodromy principle we can obtain in each so obtained simply connected domain k single-valued branches of the inverse function $z = f_l^{-1}(w)$, $l = 1, 2, \ldots, k$, where $k = n(w)$ has the same value in a given domain. We now consider the conformal metric $\varrho(w)$ which is transferred under one arbitrarily chosen branch $z = f_l^{-1}(w)$ into the logarithmic metric $-1/|z|\log r$, i.e.

$$\left(|z|\log\frac{1}{r}\right)^{-1}|dz| = \varrho(w)\,|dw|, \qquad \text{or} \qquad \varrho(w) = \left(|z|\,|f'(z)|\log\frac{1}{r}\right)^{-1}.$$

We first prove that the ϱ — length of the curves γ joining $E_f(r)$ to Γ_1 is $\geqslant 1$. We have for all γ outside a set of zero measure:

$$\int_\gamma \varrho(w)\,|dw| = \int_\gamma \left(|z|\left|\frac{dw}{dz}\right|\log\frac{1}{r}\right)^{-1}|dw| \tag{3.1}$$

$$= \int_\gamma \frac{|dz|}{|z|\log 1/r} \geqslant \int_r^1 \frac{dt}{t\log 1/r} = 1.$$

Thus, in view of the extremal property of the module, we have

$$\iint_{\Omega_r} \varrho^2(w)\,du\,dv \geqslant M\{\gamma\}. \tag{3.2}$$

Now, in view of the conformal invariance of the Dirichlet's integral, we can drop in both terms of the difference

$$A\left(\frac{-1}{|z|\log r}, B_r\right) - A\left(\varrho(w), \Omega_r\right), \qquad \Omega_r = f(B_r), \tag{3.3}$$

the areas of domains which correspond $1:1$ to each other under $f_l^{-1}(w)$. In this way we exhaust Ω_r completely, whereas the remainder left in the first term will represent the logarithmic area of the maps under some $f_l^{-1}(w)$ of these domains, where $n(w) \geqslant 2$, the maps being taken under all these $f_l^{-1}(w)$ which were not used in constructing $\varrho(w)$. In particular, the difference is not less than the logarithmic area G of the map under $f_l^{-1}(w)$ of some disk Δ contained in a domain where $n(w) \geqslant 2, f_l^{-1}(w)$ not having been used in construction of $\varrho(w)$. If we now replace r by $r_1 \in (q, r)$, the open set $\{w: n(w) \geqslant 2\}$ increases and we can obviously define in Ω_{r_1} in an analogous manner a new metric ϱ_1 choosing in all the domains overlapping Δ the same branches of $f_l^{-1}(w)$ as before. We see that the difference (3.3) will again be bounded away from 0 by G.

This means that also

$$\frac{-2\pi}{\log q} - M\{\gamma\} \geqslant G > 0$$

in the limiting case $r \to q$ and this proves Kubo's theorem.

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