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SOME CRITERIA FOR THE MULTIVALENCE OF CERTAIN ANALYTIC FUNCTIONS

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1. The following result is due to Umezawa [6]:

THEOREM (Umezawa). *Let $f(z)$ be analytic in the unit disc D: $|z| < 1$, and let $\Phi(z)$ be a p-valent convex analytic function in D. If* $\mathrm{Re}[f'(z)/\Phi'(z)] > 0$ *holds in D, then $f(z)$ is p-valent in D.*

Umezawa calls $f(z)$ in the preceding theorem a *p-valent close-to-convex function.* Such $f(z)$ are generalizations of the univalent close-to-convex functions introduced by Kaplan [3].

The following result is due to Alexander [1]:

THEOREM (Alexander). *Let $f(z) = z+a_2z^2+\ldots$ be analytic in D and let $1 \geqslant 2a_2 \geqslant 3a_3 \geqslant \ldots \geqslant 0$ hold. Then $f(z)$ is univalent and close-to-convex in D.*

The object of this note is to extend Alexander's result to the case of multivalently close-to-convex functions, and to extend certain results due to Ozaki [5]. We make use of Umezawa's result in the proofs of our theorems.

2. We can now prove the following result:

THEOREM 1. *If $f(z)$ is analytic in D, and if there exists a p-valent starlike function $\sigma(z)$ such that*

$$\mathrm{Re}\left[\frac{zf'(z)}{\sigma(z)}\right] > 0 \tag{1}$$

holds in D, then $f(z)$ is a p-valent close-to-convex function in D.

Proof. It follows from a result due to Goodman [2] that the function

$$\int_0^z \frac{\sigma(z)}{z}\,dz$$

is a p-valent convex function D. The present result now follows from Umezawa's theorem quoted above.

3. If we write $\sigma_1 \equiv z^p/(1-z)$, then a simple computation shows that

$$\operatorname{Re}\left[\frac{z\sigma_1'}{\sigma_1}\right] > \frac{1}{|1-z|^2}(1-|z|)\big(p-(p-1)|z|\big) > 0$$

holds in D. Hence σ_1 is a p-valent starlike function. For this particular function, Theorem 1 yields the following result.

THEOREM 2. *Let $f(z)$ be defined by the power series*

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \ldots, \tag{2}$$

and let

$$p \geqslant \sum_{n=p}^{\infty} |na_n - (n+1)a_{n+1}|, \qquad a_p = 1, \tag{3}$$

hold. Then $f(z)$ is analytic in D and defines a p-valent close-to-convex function there.

Proof. First we have

$$\begin{aligned} n|a_n| &= \left|\sum_{k=p}^{n-1}\big(ka_k - (k+1)a_{k+1}\big) - p\right|, \qquad a_p \equiv 1, \\ &\leqslant \sum_{k=p}^{n-1} |ka_k - (k+1)a_{k+1}| + p \\ &\leqslant \sum_{k=p}^{\infty} |ka_k - (k+1)a_{k+1}| + p \leqslant 2p. \end{aligned}$$

Hence the radius of convergence of the power series defining $f(z)$ is not less than unity. Now if we use the function σ_1 of Section 3, then for z in D we have

$$\begin{aligned} \operatorname{Re}[zf'(z)/\sigma_1(z)] &= \operatorname{Re}\left[\frac{(1-z)f'(z)}{z^{p-1}}\right] \\ &= p - \operatorname{Re}\left[\sum_{n=p}^{\infty}\big(na_n - (n+1)a_{n+1}\big)z^{n-p+1}\right], \qquad a_p \equiv 1, \\ &> p - \sum_{n=p}^{\infty} |na_n - (n+1)a_{n+1}| \geqslant 0. \end{aligned}$$

The present result now follows from Theorem 1.

COROLLARY. *Let $f(z) = z^p + a_{p+1}z^{p+1} + \ldots$ and suppose that either*

$$p \geqslant (p+1)a_{p+1} \geqslant (p+2)a_{p+2} \geqslant \ldots \geqslant na_n \geqslant \ldots \geqslant 0, \tag{4}$$

or

$$p \leqslant (p+1)a_{p+1} \leqslant (p+2)a_{p+2} \leqslant \ldots \leqslant na_n \leqslant \ldots \leqslant 2p \tag{5}$$

holds. Then $f(z)$ is a p-valent close-to-convex function in D.

4. If we use $\sigma_2 = z^p/(1-z)^2$ in Theorem 1, then we can prove the following result:

THEOREM 3. *Let $f(z) = z^p + a_{p+1}z^{p+1} + \ldots$ and let*

$$p \geqslant \sum_{n=p}^{\infty} |(n-1)a_{n-1} - 2na_n + (n+1)a_{n+1}|, \qquad a_{p-1} = 0,\ a_p = 1, \tag{6}$$

hold. Then $f(z)$ is analytic and p-valent close-to-convex in D.

COROLLARY. *Let $f(z) = z^p + a_{p+1}z^{p+1} + \ldots$ and let either*

$$p \geqslant (p+1)a_{p+1} - p \geqslant (p+2)a_{p+2} - (p+1)a_{p+1} \geqslant \ldots \geqslant 0$$

or

$$p \leqslant (p+1)a_{p+1} - p \leqslant (p+2)a_{p+2} - (p+1)a_{p+1} \leqslant \ldots \leqslant 2p$$

hold. Then $f(z)$ is analytic and p-valent close-to-convex in D.

5. If we use $\sigma_3 = z^p/(1-z^2)$ in Theorem 1, then we can prove the following assertion:

THEOREM 4. *Let $f(z) = z^p + a_{p+1}z^{p+1} + \ldots$ and let*

$$p \geqslant \sum_{n=p}^{\infty} |(n-1)a_{n-1} - (n+1)a_{n+1}|, \qquad a_{p-1} = 0,\ a_p = 1,$$

hold. Then $f(z)$ is an analytic p-valent close-to-convex function in D.

6. It would be of interest to extend the results in a recent note due to Lewandowski, Reade and Złotkiewicz to the case of p-valent functions; these authors considered variants of Alexander's theorem, but confined themselves to the case of univalent close-to-convex functions.

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