COLLOQUIUM MATHEMATICUM

VOL. XXVII 1973 FASC. 2

TWO REMARKS ON THE FOURIER-STIELTJES TRANSFORMS OF CONTINUOUS MEASURES

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1. Hartman and Ryll-Nardzewski (see [1], P 711) have asked if there exist continuous measures μ on $[0, 2\pi)$ such that

$$E(\mu, \delta) = \{n \in Z\colon |\hat{\mu}(n)| > \delta\}$$

is not a Sidon set for some $\delta > 0$. ($\hat{\mu}$ denotes the Fourier-Stieltjes transform of μ.) Kaufman [3] has shown (using Baire category) that every compact perfect set B supports a measure μ such that $E(\mu, \delta)$ is not a Sidon set for some $\delta > 0$ (1). In this note, we give a constructive proof of Kaufman's result (Corollary):

THEOREM 1. *Let $B \subseteq [0, 2\pi)$ be a perfect compact set. Then B supports a continuous measure μ such that $E(\mu, \delta)$ is not a Sidon set whenever $0 < \delta < \|\mu\|$.*

Proof. The usual construction (see 5.2.4 in [4]) shows that B contains a compact perfect Kronecker set K. Let μ be any continuous positive measure of norm one concentrated on K, and let n_j be chosen so that

$$\sup_{x \in K} |1 - e^{ixn_j}| < 2^{-2^j}, \quad j = 1, 2, \ldots, \qquad n_1 < n_2 < \ldots$$

(This is possible because K is a Kronecker set.)

An easy estimate shows that $n_j, 2n_j, \ldots, jn_j \in E(\mu, 1-2^{-j})$. Since $E(\mu, \delta) \subseteq E(\mu, \delta')$ if $\delta > \delta'$, we see that each $E(\mu, \delta)$ contains arbitrarity long arithmetic progressions and is not, therefore (see 6.8 in [4]), a Sidon set if $0 < \delta < 1$.

The family of sets $E \subseteq Z$ which have compact closure in the maximal ideal space Δ of the continuous measures on $[0, 2\pi)$ is closed under union and intersection. However, if

$$E = \{10^{10^j}\colon j = 1, 2, 3, \ldots\}$$

(1) Another solution was found by D. E. Ramirez, see this volume, p. 81-82. (Note of the Editors.)

and

$$F = \{-10^{10^j}+j\colon j = 1, 2 \ldots\} \cup \{-10^{10^j}-j\colon j = 1, 2, \ldots\},$$

then E and F have compact closures in Δ but $E+F = Z$ does not.

2. Hartman has asked (see [1] and [2]) whether there exists a continuous measure μ on $[0, 2\pi)$ and $\delta > 0$ such that $n \in Z$ implies either $\hat{\mu}(n) = 0$ or $|\hat{\mu}(n)| \geqslant \delta$, and $\hat{\mu}(n) \neq 0$ for an infinite number of n. We will prove that such measures do not exist.

LEMMA. *Let $E_\mu = \{n \in Z\colon |\hat{\mu}(n)| \geqslant \delta\}$, where μ is a continuous measure. Then E has arbitrarily long gaps, that is, if $E = \{n_j\}_{-\infty}^{\infty}$ and $n_j < n_{j+1}$, then*

$$\sup_{1<j<\infty} (n_{j+1} - n_j) = \infty.$$

Proof. Otherwise, there exists $C > 0$ such that

$$\sup_{1\leqslant i<\infty} (n_{j+1} - n_j) \leqslant C,$$

so

$$\frac{1}{n_j - n_1} \sum_{n=n_1}^{n_j} |\hat{\mu}(n)|^2 \geqslant \frac{\delta^2}{C}.$$

This contradicts the known result of Wiener [4], Chapter 5.

THEOREM 2. *Let μ be a continuous measure on $[0, 2\pi)$ and $\delta > 0$. Suppose that $|\hat{\mu}(n)| \geqslant \delta$ for an infinite number of n. Then, for some n, $0 < |\hat{\mu}(n)| < \delta$.*

Proof. We argue by contradiction. By the Lemma, we know that

$$E = \{n\colon |\hat{\mu}(n)| \geqslant \delta\} = \{n_j\}_{-\infty}^{\infty}$$

is such that

$$\sup_{1\leqslant j<\infty} (n_{j+1} - n_j) = \infty.$$

Choose $j_1, j_2, \ldots$ such that

(1) $$n_{j_k+1} - n_{j_k} \geqslant k \quad \text{for } k = 1, 2, \ldots$$

It is clear that if $d\mu_k = \exp(-in_{j_k}x)\,d\mu$, $\{\mu_k\}$ contains a subsequence which converges weak $*$ to a measure ν on $[0, 2\pi)$. So we can suppose

(2) $$\lim_k \hat{\mu}(m + n_{j_k}) = \hat{\nu}(m) \quad \text{for all integers } m.$$

We then have

(a) ν is singular (a result of Helson (3.5.1 in [4])), and $|\hat{\nu}|(0) \geqslant \delta$,
(b) $\hat{\nu}(m) = 0$ if $m > 0$ (by (1) and (2)), so
(c) $\nu \in L^1(0, 2\pi)$ (by the F. and M. Riesz Theorem (see 8.2.1 in [4])).

The contradiction of (a) and (c) shows that $0 < |\hat{\mu}(n)| < \delta$ for some n. (Professor Hartman suggested using Helson's result and remarked that actually more is proved, namely that $\lim_{\substack{n\to\infty,\\ n\notin E}} \hat{\mu}(n) = 0$ is impossible.)

REFERENCES

[1] S. Hartman, *P 695*, Colloquium Mathematicum 22 (1970), p. 158.

[2] — and C. Ryll-Nardzewski, *Quelques résultats et problèmes en algèbre des mesures continues*, ibidem 22 (1971), p. 271-277.

[3] R. Kaufman, *Remark on Fourier-Stieltjes transforms of continuous measures*, ibidem 22 (1971), p. 279-280.

[4] W. Rudin, *Fourier analysis on groups*, New York - London 1962.

Reçu par la Rédaction le 3. 8. 1971;
en version modifiée le 20. 2. 1972