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ITERATED FAMILIES

BY

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In what follows I propose to sketch the chapter in "Analysis without epsilons" (or "Topology without open sets") dealing with extensions of continuous functions on dense subspaces.

As carrier of the convergence structure I adopt a common generalization of sequence and filter which I call *family* (*on* X) (1): it is a triple consisting of an auxiliary set T, a collection $\mathscr{F}$ of its subsets, and a function x from T into the space X in question. This will be written simply $\mathscr{F}$, the underlying set T and associated function x being understood. I come back to well-known notions if x is the identity or if T is partially ordered and $\mathscr{F}$ the collection of final subsets; although I have not imposed the usual directedness requirement since it is nowhere needed (2). Actually I have no interest in T, and so agree to regard its replacement by any one-one image (with preservation of $\mathscr{F}$ and x of course) as not changing the family.

A central role will be played by the following construct [7]: Let $\mathscr{F}$ be a family on X and for each $t \in T$ let $\mathscr{F}(t)$ be a family on X, underlying sets being taken disjoint. The *iterated family* $\mathscr{F}(t):\mathscr{F}$ has as underlying set the union of those underlying the $\mathscr{F}(t)$; as associated function the (unique) one extending the functions associated to the $\mathscr{F}(t)$; and as collection the sets $\bigcup\{A(t): t \in A\}$, where $A(t) \in \mathscr{F}(t)$ and $A \in \mathscr{F}$ are chosen in all possible ways.

To treat convergence and uniform convergence in a single pattern I introduce the idea of a *convergence on* X: this is just a non-void set C of families on X called *convergent*. Enlarging C makes the convergence *stronger*.

If f is a function on X it turns every family $\mathscr{F}$ on X into a family, written $f(\mathscr{F})$, on its range: T and $\mathscr{F}$ remain unchanged while the asso-

(1) This notion is quite well known: see, for example, the introduction to Bourbaki's *Topologie générale* [4].

(2) On the other hand, one can if one wishes read all $\mathscr{F}$ as filter bases since none of the constructions lead out of this class.

ciated function is composed with f. If it respects the convergences present I call f *continuous*.

I propose to study the interplay on X of a convergence C^* with another C' (interpretable as C^* restricted to, i.e. consisting of those families whose functions map into, a subspace) and with convergences $C(x)$ indexed by X (interpretable, in the presence of additional structure, as the families convergent to x): I shall say that C^* is an $\mathscr{I}$ *convergence* ($\mathscr{R}$ *convergence*) with respect to C', $C(x)$ if $\mathscr{F} \in C^*$ implies (is implied by) $\mathscr{F}(t): \mathscr{F} \in C'$ for every (some) choice of $\mathscr{F}(t) \in C\big(x(t)\big)$.

There is a strongest $\mathscr{I}$ and a weakest $\mathscr{R}$ convergence, the latter being stronger than the former. Moreover, the convergence C^* restricted to a subspace X' is $\mathscr{I}$ or $\mathscr{R}$ with respect to C' and $C(x)$ so restricted; while if C' and the $C(x)$ are on X', then the restriction of an extremal convergence is the extremal convergence on X'. Note that C' lies between the extremal convergences if $\big\{\{x\}\big\} \in C(x)$ for every $x \in X$: because iteration with $\mathscr{F}(t) = \big\{\{t\}\big\}$ leaves a family unchanged.

Somewhat deeper is: If $C(x_0)$ is, for each $x_0 \in X$, an $\mathscr{I}$ convergence with respect to $C'(x_0)$, $C'(x)$, then the strongest $\mathscr{I}$ (weakest $\mathscr{R}$) convergence C^* with respect to C', $C'(x)$ is an $\mathscr{I}(\mathscr{R})$ convergence with respect to C^*, $C(x)$. The proof turns on the associativity of iteration.

Let $f: X \to Y$ be continuous for the convergences C' on X and Y, and satisfy $f\big(C(x)\big) \subset C\big(f(x)\big)$ for every $x \in X$. Then f is continuous for C^* if on Y it is an $\mathscr{R}$ convergence and on X the weakest $\mathscr{R}$ convergence (or of course any weaker, say one of the $\mathscr{I}$ convergences). Indeed if $\mathscr{F} \in C^*$ on X, then for some $\mathscr{F}(t) \in C\big(x(t)\big), \mathscr{F}(t): \mathscr{F} \in C'$; therefore $f\big(\mathscr{F}(t): \mathscr{F}\big) = f\big(\mathscr{F}(t)\big): f(\mathscr{F}) \in C'$ on Y which, in view of $f\big(\mathscr{F}(t)\big) \in C\Big(f\big(x(t)\big)\Big)$, yields $f(\mathscr{F}) \in C^*$.

It remains to exhibit all this in the context of neighborhoods. I shall say of families on X that $\mathscr{F}$ is *cofinal in* $\mathscr{F}'$, written $\mathscr{F} < \mathscr{F}'$, if the image of every set in $\mathscr{F}'$ contains the image of some set in $\mathscr{F}$; and that a convergence is of *neighborhood type* if it consists of all (3) families cofinal in some *neighborhood family* $\mathscr{U}$ (for which I may take $T = X$ and x the identity). Besides containing every family cofinal in one of its families (4), such a C contains, with $\mathscr{F}(t)$ for all $t \in T$, also $\mathscr{F}(t): \{T\}$. Conversely, a convergence enjoying such closure properties is of neighborhood type: $\mathscr{U}$ is just the image in X of $\mathscr{F}(t): \{T\}$ with T the set of families in C whose associated function is the identity.

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(3) In order to have C a set it is necessary to make some restriction, say of cardinality, on the allowed T's.

(4) This condition is necessary and sufficient for C to be a union of neighborhood convergences.

If C' is of neighborhood type and each $C(x)$ is closed with respect to $\mathscr{F}(t)\colon \{T\}$ (5), then the strongest $\mathscr{I}$ convergence is of neighborhood type. Indeed, if $\mathscr{F} < \mathscr{F}' \in C^*$, given $\mathscr{F}(t) \in C(x(t))$ for $t \in T$, take $\mathscr{F}(t')$ as $\mathscr{F}(t)\colon \{x^{-1}x'(t')\}$ for $t' \in x'^{-1}x(T)$ and arbitrarily in $C(x'(t'))$ otherwise; then $\mathscr{F}(t)\colon \mathscr{F} < \mathscr{F}(t')\colon \mathscr{F}'$, therefore $\mathscr{F} \in C^*$. The remaining property follows again from the associativity of iteration.

If C^*, C', and $C(x)$ are all of neighborhood type with respective neighborhood families $\mathscr{U}^*$, $\mathscr{U}'$, $\mathscr{U}(x)$, then C^* is an $\mathscr{I}$ convergence if and only if $\mathscr{U}(x)\colon \mathscr{U}^* < \mathscr{U}'$. Indeed, taking $T = X$, $\mathscr{F} = \mathscr{U}^*$, and $\mathscr{F}(x) = \mathscr{U}(x)$ shows the condition necessary; conversely, $\mathscr{F} < \mathscr{U}^*$ implies $\mathscr{F}(t)\colon \mathscr{F} < \mathscr{U}(x)\colon \mathscr{U}^* < \mathscr{U}'$.

Again, if C^* and C' are of neighborhood type, C^* is an $\mathscr{R}$ convergence if and only if $\bar{\mathscr{U}}' < \mathscr{U}^*$ (6). Here $x \in \bar{A}$ means $\mathscr{F} < \{A\}$ for some $\mathscr{F} \in C(x)$. Indeed, from $\mathscr{F}(t)\colon \mathscr{F} < \mathscr{U}'$ follows $\mathscr{F} < \bar{\mathscr{U}}'$ proving sufficiency; conversely, if $\bar{\mathscr{U}}' \notin C^*$, let $T = \{(y, U')\colon y \in \bar{U}'\}$, $x(y, U') = y$, $\mathscr{F} = \{\{y, U')\colon U' \subset V'\}\colon V' \in \mathscr{U}'\}$, and $\mathscr{F}(y, U')$ be any family $< \{U'\}$ in $C(y)$; then $\mathscr{F}(y, U')\colon \mathscr{F} < \mathscr{U}'$ but $\mathscr{F} \notin C^*$.

By a *convergence space* I shall mean a set X equipped with an assignment to each of its points x of a convergence $C(x)$, these families being called *convergent to* x. A function f is now to be continuous if $f(C(x)) \subset C(f(x))$; while C^* makes X into an $\mathscr{I}$ or $\mathscr{R}$ *convergence space* with respect to C' and C (in that order) if, for every $x_0 \in X$, $C^*(x_0)$ is an $\mathscr{I}$ or $\mathscr{R}$ convergence with respect to $C'(x_0)$ and $C(x)$.

Little will be gained by writing out the straightforward specialization to convergence spaces of the above development. I content myself with noting that besides the correspondence of C^* with C', either or both of these could now be confronted with C; and with citing those results for which I know an analogue in the literature.

Let X be an overset of the convergence space X' and let $C'(\ \ )$ be extended to the $x \in X$. Then the strongest $\mathscr{I}$ convergence space C^* on X with respect to C' (and $C = C'$) is an $\mathscr{I}$ convergence with respect to C^* (and $C = C^*$). This was proved by Kowalsky [8] under additional assumptions, among them that $\{\{x'\}\} \in C'(x')$ and that C' be an $\mathscr{I}$ convergence — therefore the strongest $\mathscr{I}$ convergence — with respect to itself, since he wants to conclude that C' is C^* restricted to X'; and that the $C'(x)$ be distinct for different x, since he wants to describe X in terms of them (a requirement dropped in the fifth chapter of [9], where however only topological spaces are treated). Note that if the $C'(x)$ are *disjoint* for different x, then in any $\mathscr{I}$ convergence space limits are unique.

(5) This is the case if $C(x)$ is an $\mathscr{I}$ convergence with respect to itself on the one point space $\{x\}$.

(6) More generally, for C^*, C' unions of neighborhood convergences, if and only if $\bar{C}' \subset C^*$.

Let X be a convergence space with $C(x)$; X' a subset such that $C'(x)$, which is $C(x)$ restricted to X', is never void ("X' is dense in X"); f a function on X to the convergence space Y continuous for C'("f is continuous on X' and is extended to X by continuity"). Then f is continuous for C. The result was proved by Bourbaki and Dieudonné [5] (and appears in [4]) for the case that X and Y are topological spaces; and subsequently generalized by Appert [1] to the case that Y is a neighborhood type space.

It will be recalled that convergence in a topological space is specified by neighborhood filter bases $\mathscr{U}(x)$ satisfying $\mathscr{U}(x)\colon \mathscr{U}(x_0) < \mathscr{U}(x_0)$; and that $\overline{\mathscr{U}(x_0)} < \mathscr{U}(x_0)$ characterizes the regular spaces. For this case the equivalences announced go back to Birkhoff [2]; for a particularly lavish treatment see Grimeisen [7].

By a *uniform convergence space* I shall understand a set X equipped with a convergence on $X \times X$, the convergent families now being called *uniformly convergent* (7) and the functions on X whose extensions to $X \times X$ preserve them *uniformly continuous.*

One can make correspond to each uniform convergence space an (ordinary) convergence space by taking for the families convergent to x, for example, the projections on the first factor of those uniformly convergent families whose other projection is $\{\{x\}\}$ (on the assumption there are such for every x). Then every uniformly continuous function is continuous and every convergence space can be "uniformized" with a weakest convergence.

Given quite generally convergences $C(x)$ on the uniform convergence space X, I propose for the $C(x, y)$ with respect to which uniform $\mathscr{I}$ and $\mathscr{R}$ convergences are to be defined the products (defined in the obvious way) of an element of $C(x)$ and one of $C(y)$ (8). If $\{\{x\}\} \in C(x)$, then there corresponds to a uniform $\mathscr{I}(\mathscr{R})$ convergence an ordinary $\mathscr{I}(\mathscr{R})$ convergence. The proof hinges on passing from $\mathscr{F}(t) \in C\big(x(t)\big)$ to $\mathscr{F}(t) \times \big\{\{x\}\big\} \in C\big(x(t), x\big)$.

I shall again restrict myself to translating only a few of the general considerations to this setting: Assuming C^*, C', and $C(x, y)$ of neigh-

(7) Not to be confused with the convergence, uniform in λ, of a set $(T, \mathscr{F}, x^\lambda)$ of families in a convergence: In case the convergence criterion is couched in terms of the image of the family in the space (as, for example, for neighborhood type) the latter may be defined as the convergence of $\{\bigcup_\lambda x^\lambda(A) : A \in \mathscr{F}\}$

(8) In order to have the $C(x, y)$ of neighborhood type when the $C(x)$ are, one should instead take the families on $X \times X$ whose projections belong to $C(x)$ and $C(y)$ respectively, i.e. the strongest convergence space on $X \times X$ making the projections continuous (for filter bases these are just the families cofinal in some product): indeed, $\mathscr{U}(x) \times \{X\} \cup \{X\} \times \mathscr{U}(y)$ (for filter bases $\mathscr{U}(x) \times \mathscr{U}(y)$) is a neighborhood family for $C(x, y)$. What follows can be carried through for this choice of $C(x, y)$ with only the slightest modification.

borhood type, C^* is a uniform $\mathscr{I}$ convergence if and only if $\mathscr{U}(x, y)$: $\mathscr{U}^* < \mathscr{U}'$; a uniform $\mathscr{R}$ convergence if and only if $\overline{\mathscr{U}}' < \mathscr{U}^*$. The left sides are respectively the neighborhoods in $X \times X$ equipped with the $C(x, y)$ of the $\mathscr{U}^*$ (if one defines a neighborhood of a set to be any union of neighborhoods of each of its points) and the closures of the $\mathscr{U}'$. More particularly, if $\{\{(x, y)\}\} < \mathscr{U}(x, y)$, a uniform convergence on X is an $\mathscr{I}$ or $\mathscr{R}$ convergence with respect to itself according as the neighborhoods or the closures of its uniform neighborhoods generate its uniform convergence. Davis [6] has shown that the convergence of a topological space corresponds to the uniform convergence defined by the family of open neighborhoods of the diagonal in the product space exactly when open sets contain the closures of their points. Similarly, the convergence of a regular topological space corresponds to the uniform convergence of closed neighborhoods of the diagonal in the product space. Indeed, given $x_0 \in U$ open, there exist open V, W with $x_0 \in W \subset \overline{W} \subset V \subset \overline{V} \subset U$; then $X \times (X - W) \cup \overline{V} \times X$ is a closed neighborhood of the diagonal (if $x \notin V$, $x \in X - \overline{W} \subset X - W$) which intersects $X \times \{x_0\}$ in $\overline{V} \times \{x_0\}$.

The translation: If f is uniformly continuous for the C' on a uniform $\mathscr{I}$ convergence X to a uniform $\mathscr{R}$ convergence Y and $f(C(x)) \subset C(f(x))$ (i.e. f is continuous for the ordinary convergence defined by the C), then f is uniformly continuous for the C^*. This yields the classical result when C' is C^* restricted to a subspace, except for the possibility of extending f, if initially defined only on the subspace, so as to satisfy the hypothesis.

To accomplish this I shall assume $\{\{(x, x)\}\} \in C^*$ for every $x \in X$, whence every family both of whose projections belong to $C(x)$ is an element of C' (9). This is true in particular of $\mathscr{F} \times \mathscr{F}'$ with $\mathscr{F}, \mathscr{F}' \in C(x)$; setting $\mathscr{F} = \mathscr{F}'$ yields something appropriately called the Cauchy condition; while if conversely every Cauchy family belongs to some $C(x)$, C' may be called *complete*.

Parenthetically, the completeness of a completion follows from: Let C^* be a uniform $\mathscr{I}$ convergence with respect to C', $C(x)$ and let $C^*(x_0)$ be an ordinary $\mathscr{R}$ convergence with respect to $C(x_0)$, $C(x)$. If every C' Cauchy family belongs to some $C^*(x)$, then so does every C^* Cauchy family. Indeed, if $\mathscr{F}$ is C^* Cauchy choose any $\mathscr{F}(t) \in C(x(t))$; then $\mathscr{F}(t) \times \mathscr{F}(t'): \mathscr{F} \times \mathscr{F} = \mathscr{F}(t): \mathscr{F} \times \mathscr{F}(t): \mathscr{F} \in C'$, $\mathscr{F}(t): \mathscr{F} \in C^*(x)$, $\mathscr{F} \in C^*(x)$.

Now a uniformly continuous function preserves Cauchy families: therefore if Y is complete, $\mathscr{F} \in C(x)$ will imply $f(\mathscr{F}) \in C(y)$ for some y. Moreover $\mathscr{F}' \in C(x)$ implies $f(\mathscr{F}) \times f(\mathscr{F}') \in C'$; if from this one could conclude $f(\mathscr{F}') \in C(y)$, then, with $y = f(x)$, f is extended as desired. This will be so for example if $\mathscr{F} \times \mathscr{F}' \in C'$ is symmetric and transitive for families on Y and $C(y)$ is the ordinary convergence corresponding to C'.

(9) In the strongest $\mathscr{I}$ convergence the converse holds as well. The alternate definition of footnote (8) is being used here.

If convergence is a function only of the image family in the space, transitivity follows from the closure of C' with respect to (the extension to families of) relational composition (written $\circ$ by Bourbaki). In this case previous criteria can also be given an alternative form in view of ([10]) $x(\mathscr{F}(t)):\mathscr{F} = (x\mathscr{F}(t)\times\{t\}):\{T\}\circ\mathscr{F}$ (here $\circ$ maps $\mathscr{P}(X\times T)\times\mathscr{P}(T)$ into $\mathscr{P}(X)$); for uniform convergence spaces the identity

$$[\pi_x\mathscr{F}(t)\times\pi_y\mathscr{F}(t)]:\mathscr{F} = (\pi_x\mathscr{F}(t)\times\{t\}):\{T\}\circ\mathscr{F}\circ(\{t\}\times\pi_y\mathscr{F}(t)):\{T\}$$

should be used. Moreover,

$$x(\mathscr{F}'(t)):\mathscr{F} = (x\mathscr{F}'(t)\times x\mathscr{F}(t)):\{T\}\circ x(\mathscr{F}(t)):\mathscr{F}$$

and

$$\begin{aligned}(\pi_x\mathscr{F}'(t)\times\pi_y\mathscr{F}'(t)):\mathscr{F} \\ = (\pi_x\mathscr{F}'(t)\times\pi_x\mathscr{F}(t)):\{T\}\circ\mathscr{F}(t):\mathscr{F}\circ(\pi_y\mathscr{F}(t)\times\pi_y\mathscr{F}'(t)):\{T\}\end{aligned}$$

show that the weakest $\mathscr{R}$ and strongest $\mathscr{I}$ (uniform, or corresponding ordinary) convergences coincide if for example in addition to the above hypotheses C' is of neighborhood type and the $C(x)$ consist of Cauchy families.

In conclusion I shall treat an application to the completion of abstract metric spaces developed jointly with R. De Marr.

For families on a complete lattice Ω,

$$\limsup_{\mathscr{F}}(\limsup_{\mathscr{F}(t)}\omega) \leqslant \limsup_{\mathscr{F}(t):\mathscr{F}}\omega;$$

therefore the convergence to the smallest element 0 in Ω is $\mathscr{R}$ with respect to itself and the $C(\omega)$ of those $\mathscr{F}$ whose lim sup is ω ([11]). It follows that an Ω-valued function ϱ, continuous for C' on X, will be continuous for the weakest $\mathscr{R}$ convergence C^* with respect to convergences $C(x)$ for whose elements

$$\limsup_{\varrho(\mathscr{F})}\varrho = \varrho(x).$$

In particular, a continuous ϱ extended to an overset by this equation (of course $\limsup_{\varrho(\mathscr{F})}$ must then have the same value for all $\mathscr{F}\in C(x)$) will be continuous for the weakest $\mathscr{R}$ convergence.

For a ϱ satisfying $\varrho(C(x))\subset C(\varrho(x))$ the convergence induced by ϱ (i.e. the strongest for which it is still continuous, is $\mathscr{I}$ or $\mathscr{R}$ according

([10]) Throughout $\{T\}$ may be replaced by $\mathscr{F}$ if the latter is a filter base.

([11]) De Marr has proved a partial converse. Since $\{\{\omega\}\}\in C(\omega)$ it is even weakest $\mathscr{R}$.

as the convergence in the image space is. The convergence to 0 in Ω will be $\mathscr{I}$ provided the infinite distributive law

$$\bigvee_t \bigwedge_{\mathscr{F}(t)} \omega(t, A) = \bigwedge_{\times_t \mathscr{F}(t)} \bigvee_t \omega\big(t, A(t)\big)$$

holds ([2], p. 146, Eq. (22′); this is always the case for a linearly ordered set), at least cofinally in $\mathscr{F}$ whenever

$$\limsup_{\mathscr{F}} \bigwedge_{\mathscr{F}(t)} \omega(t, A) = 0.$$

In this case the convergence of an overset which is both $\mathscr{I}$ and $\mathscr{R}$ with respect to the convergence induced by ϱ on the subset is itself induced by the extended ϱ.

More particularly, let Ω be equipped with a binary operation $+$ which is commutative, has 0 as unit, and for which

$$\limsup_{\mathscr{F}\times\mathscr{F}'} \big(\omega(t)+\omega'(t')\big) \leqslant \limsup_{\mathscr{F}} \omega(t) + \limsup_{\mathscr{F}} \omega'(t'),$$

at least when one of the summands on the right is zero; and let ϱ be a function of two variables (thus making X into a uniform convergence space) satisfying the usual metric axioms. If the projections of the family $\mathscr{F}$ on $X\times X$ converge for the underlying ordinary convergence to x and x' respectively, then

$$\varrho(x, x') = \limsup_{\mathscr{F}} \varrho;$$

this justifies the choice of $C(\omega)$ above.

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