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THERE ARE ABSOLUTE ULTRAFILTERS ON N WHICH ARE NOT MINIMAL

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In our previous paper [6] we have distinguished absolute P points in $N^* = \beta N \setminus N$ and we have proved that a known consequence of Martin's Axiom (MA), called *Booth's Lemma* (BL), implies the existence of $2^{\mathfrak{c}}$ points in N^* which are both absolute and minimal in the sense of Rudin-Keisler (RK) ordering. The aim of this paper is to show that there are $2^{\mathfrak{c}}$ absolute points in N^* which are not minimal. We also prove Rothberger's Lemma which is used in the proof of the main theorem.

1. Preliminaries. In what follows N denotes the set of positive integers with the discrete topology. The remainder N^* of the Čech-Stone compactification of N consists of free ultrafilters on N. An ultrafilter $x \in N^*$ is said to be *absolute* (*absolute P-point* of N^*) if it has a base linearly ordered by the relation: $A < B$ iff $A \setminus B$ is finite and $B \setminus A$ is infinite. An ultrafilter $x \in N^*$ is said to be an $\mathfrak{m}$-*ultrafilter* if for any subfamily R of x of cardinality less than $\mathfrak{m}$ there exists an element B of x such that $B < A$ for each A from R; $\aleph_1$-ultrafilters are usually called *P-points* of N^*, and absolute points are $\mathfrak{c}$-*ultrafilters* in this terminology. Let us note that there exist $\mathfrak{m}$-ultrafilters on N which are not $\mathfrak{m}^+$-ultrafilters for each regular $\mathfrak{m}$, $\aleph_0 < \mathfrak{m} < \mathfrak{c}$, whenever MA holds (see [9] and [10]). The number $\mathfrak{c}$ is the greatest number $\mathfrak{m}$ for which $\mathfrak{m}$-ultrafilters can exist on N. Another description of absolute ultrafilters can be stated in terms of $\mathfrak{c}$-*towers* (Hechler [5]), i.e., of the decreasing families (in the sense of the relation $<$ introduced above) of cardinality $\mathfrak{c}$; an *absolute ultrafilter* is an ultrafilter having a $\mathfrak{c}$-tower as a base. An ultrafilter $x \in N^*$ is said to be *RK-minimal* if it is a P-point and if for any finite-to-one map $f\colon N \to N$ there exists a member A of x such that $f|A$ is one-to-one. Blass proved [1] that MA implies the existence of $2^{\mathfrak{c}}$ points in N^* which are minimal, and the existence of P-points which are not minimal. In this paper we use only the following consequence of MA (see Booth [2]):

BL. *If $\mathscr{F}$ is a filterbase of fewer than $\mathfrak{c}$ infinite subsets of N, then there exists an infinite subset T of N such that $T < A$ for $A \in \mathscr{F}$.*

As claimed by Kunen and Tall [7], BL is essentially weaker than MA.

2. Generalization of Rothberger's lemma. The lemma which follows is a generalization of a known Rothberger's lemma [8]. This lemma says that the non-existence of ω_1-limits implies the non-existence of (ω_1, ω_0)-gaps on N if we pass to the terminology which is widely used (see Tall [11] or Engelking [4] for the review of results).

LEMMA 1 (BL). *If $\mathscr{F}$ is a filterbase on N such that* $\operatorname{card}\mathscr{F} < \mathfrak{c}$ *and $\{H_1, H_2, \ldots\}$ is a sequence of infinite subsets of N such that $H_n < F$ for $n \in N$ and $F \in \mathscr{F}$, then there exists an infinite subset B of N such that $H_n < B < F$ for $n \in N$ and $F \in \mathscr{F}$.*

Proof. Without loss of generality we may assume that sets H_n are mutually disjoint. Let $\{F_\beta\colon \beta < a\}$, $a < \mathfrak{c}$, be a well ordering of $\mathscr{F}$. We define, using BL, a free filterbase $\{T_\beta\colon \beta < a\}$ on N so that, for any β, $\beta < a$, and for any $n \in N$,

(*) the k-th member of H_n belongs to F_β whenever k is greater than the n-th member of T_β.

The m-th member of a subset A of N is the m-th member of A in the natural order of N.

We proceed by transfinite induction.

Since $H_n \setminus F_0$ is finite, there exists a t_n^0 in N such that the k-th member of H_n belongs to F_0 whenever k is greater than t_n^0. We can choose these t_n^0's to form an increasing sequence. Let $T_0 = \{t_1^0, t_2^0, \ldots\}$.

Suppose that T_γ for $\gamma < \beta$, where $\beta < a$, are already defined. From BL it follows that there exists an infinite subset A of N such that $A < T_\gamma$ for $\gamma < \beta$. Since $H_n \setminus F_\beta$ is finite, there exists a $t_n^\beta \in A$ such that the k-th member of H_n belongs to F_β whenever k is greater than t_n^β. We can choose these t_n^β's to form an increasing sequence. Let $T_\beta = \{t_1^\beta, t_2^\beta, \ldots\}$. Clearly, T_β satisfies (*) and $\{T_\beta\} \cup \{T_\gamma\colon \gamma < \beta\}$ is again a filterbase, T_β being an infinite subset of A, and $A < T_\gamma$ for all $\gamma < \beta$.

Thus the filterbase $\{T_\beta\colon \beta < a\}$ is defined.

Applying BL to that filterbase we get an infinite subset T of N such that $T < T_\beta$ for $\beta < a$. Since sets $T \setminus T_\beta$ are finite, the $2n$-th element of T is greater than the n-th element of T_β for all but finitely many n from N.

Let B_k be a subset of H_k consisting of elements beginning with the $m(k)$-th one, where $m(k)$ is equal to the $2k$-th element of T. Let

$$B = \bigcup \{B_k\colon k \in N\}.$$

For each $n \in N$, sets $H_n \setminus B$ and $H_n \setminus B_n$ are equal and are finite (H_n's being mutually disjoint). Therefore, $H_n < B$ for each $n \in N$.

For each β, $\beta < \alpha$, there exists a $k(\beta) \in N$ such that B_k is contained in F_β whenever $k \geqslant k(\beta)$. To get such a $k(\beta)$, let us recall that B_k consists of elements of H_k beginning with the $m(k)$-th element, where $m(k)$ is the $2k$-th member of T. In view of $(*)$ these elements are in F_β whenever $m(k)$ is greater than the k-th member of T_β. However, all but finitely many $2k$-th members of T are greater than the k-th elements of T_β. We choose $k(\beta)$ to be such that, for $k \geqslant k(\beta)$, the $2k$-th member of T is greater than the k-th member of T_β and this number is the desired one. We have $B_k < F_\beta$ for $k \in N$, since $B_k \subset H_k$ and $H_k < F_\beta$. Thus $\bigcup \{B_k\colon k < k(\beta)\} \setminus F_\beta$ is finite for each β. Since $B_k \setminus F_\beta = \emptyset$ for $k \geqslant k(\beta)$, $B \setminus F_\beta$ is finite. This means that $B < F_\beta$.

3. Lemmas on large filterbases. For the purpose of our main theorem we consider N to be a union of disjoint and finite subsets, $N = X_1 \cup X_2 \cup \ldots$, such that the sequence $\{\operatorname{card} X_n\colon n \in N\}$ tends to infinity. Let us fix such a decomposition of N. Let $f\colon N \to N$ be the map assigning to each element of X_n the number n. In order to avoid misunderstanding we denote by X the copy of N being the domain of f. Thus we have $X = X_1 \cup X_2 \cup \ldots$ We construct on X an ultrafilter x which has a $\mathfrak{c}$-tower $\mathcal{F}$ as a base (i.e., which is absolute) and such that for each $A \in \mathcal{F}$ the sequence $\operatorname{card}(A \cap X_n)$ tends to infinity if n runs over an infinite subset $B(A)$ of N. In view of the last property, the map $f\colon X \to N$ cannot be one-to-one on elements of x. Construction of the tower $\mathcal{F}$ goes by induction over ordinals less than $\mathfrak{c}$, and in each step of the induction, passing from α to $\alpha+1$, we can construct the $(\alpha+1)$-st member of $\mathcal{F}$ in at least two ways. Thus we get $2^{\mathfrak{c}}$ of such points.

For the brevity, a subset X' of X will be called *large* if

$$\limsup \operatorname{card}(X' \cap X_n) = \infty$$

and it will be called *large relatively to* A, where A is an infinite subset of N, if

$$\lim_{n \in A} \operatorname{card}(X' \cap X_n) = \infty.$$

A filterbase $\mathcal{F}$ on X will be called *large* if each $F \in \mathcal{F}$ is large relatively to some infinite subset of N.

LEMMA 2 (BL). *If $\mathcal{F}$ is a large filterbase on X such that* $\operatorname{card}\mathcal{F} < \mathfrak{c}$, *then there exists an infinite subset B of N such that each $F \in \mathcal{F}$ is large relatively to B.*

Proof. For any $F \in \mathcal{F}$ and any $k \in N$, consider the subset $B_{k,F}$ of N consisting of those n for which $\operatorname{card}(F \cap X_n) > k$. Since $\mathcal{F}$ is large, the sets $B_{k,F}$ form a filterbase, and since $\operatorname{card}\mathcal{F} < \mathfrak{c}$, there exist less than $\mathfrak{c}$ elements in that base. Applying BL, we get an infinite subset B of N such that $B < B_{k,F}$ for any $B_{k,F}$. This means that $\lim \operatorname{card}(F \cap X_n) = \infty$ whenever n runs over B, i.e., F is large relatively to B.

LEMMA 3 (BL). *If $\mathcal{F}$ is a free filterbase on X such that* $\operatorname{card}\mathcal{F} < \mathfrak{c}$, *then there exist an infinite subset B of N and a sequence $x = \{x_n\colon n \in B\}$, $x_n \in X_n$, such that $x < F$ for each $F \in \mathcal{F}$.*

Proof. Applying BL to the filterbase $\mathcal{F}$ we get an infinite subset C of X such that $C < F$ for all $F \in \mathcal{F}$. Let

$$B = \{n \in N\colon C \cap X_n \neq \emptyset\}.$$

In each $C \cap X_n$, where $n \in B$, take a point x_n. The sequence $x = \{x_n\colon n \in B\}$ and the set B satisfy the lemma.

LEMMA 4 (BL). *If $\mathcal{F}$ is a large filterbase on X such that* $\operatorname{card}\mathcal{F} < \mathfrak{c}$, *then there exists an infinite subset T of X such that $T < F$ for all $F \in \mathcal{F}$, and there exists an infinite subset B of N such that T is large relatively to B. In particular, $\mathcal{F} \cup \{T\}$ is a large filterbase on X.*

Proof. Applying preceding lemmas, we define, by induction, sequences $x^1, x^2, \ldots$ and infinite subsets $B_1, B_2, \ldots$ of N such that

(1) $x^k\colon B_k \to X$,

(2) $x^k(n) \in X_n$,

(3) $B_1 \supset B_2 \supset \ldots$,

(4) the sequences x^k are mutually disjoint,

(5) $x^k < F$ for each $k \in N$ and for each $F \in \mathcal{F}$,

(6) each $F \in \mathcal{F}$ is large relatively to B_1.

To get x^1 and B_1 we first apply Lemma 2 to $\mathcal{F}$ and get an infinite subset B' of N such that each $F \in \mathcal{F}$ is large relatively to B'. Next, applying Lemma 3 to

$$\mathcal{F}' = \big\{F \cap \bigcup \{X_k\colon k \in B'\}\colon F \in \mathcal{F}\big\}$$

we get the desired infinite subset B_1 of B' and a sequence $x^1 = \{x_n\colon n \in B_1\}$.

Assume that the sequences $x^1, x^2, \ldots, x^n$ and infinite sets $B_1, \ldots, B_n$ of N are defined in such a way that conditions (1)-(6) are satisfied. By Lemma 3 applied to

$$X^n = \bigcup \{X_k\colon k \in B_n\} \setminus (x^1 \cup x^2 \cup \ldots \cup x^n),$$

instead of to X, and to the large filterbase $\{F \cap X^n\colon F \in \mathcal{F}\}$ on X^n we get an infinite subset B_{n+1} of B_n and a sequence $x^{n+1}\colon B_{n+1} \to X^n$ such that $x^{n+1}(m) \in X_m \setminus (x^1 \cup \ldots \cup x^n)$ for $m \in B_{n+1}$ and such that $x^{n+1} < F \cap X^n$.

Thus the sets $B_1 \supset B_2 \supset \ldots$ and the sequences $x^1, x^2, \ldots$ are defined. We have $x^k < F$ for each $k \in N$ and $F \in \mathcal{F}$. By Lemma 1, we get a subset T of X such that $x^k < T < F$ for each $k \in N$ and $F \in \mathcal{F}$. The set T is large relatively to an infinite subset B of N, which is almost contained in every B_i.

For each $k \in N$ there exists $m \in N$ such that $x^i(n) \in T$ if $n \in B$, $n > m$, and $i = 1, \ldots, k$. So $\operatorname{card} T \cap X_n \geqslant k$ for $n \in B$ and $n > m$.

MAIN THEOREM (BL). *There exist $2^{\mathfrak{c}}$ ultrafilters on N which are absolute but not minimal.*

Proof. Let $X = X_1 \cup X_2 \cup \ldots$ be a decomposition of the set X of positive integers into finite sets such that $\lim \operatorname{card} X_n = \infty$. Let $f\colon X \to N$ be the (finite-to-one) map assigning to each element of X_n the number n. As sketched in Section 2, the proof consists on a construction of $2^{\mathfrak{c}}$ $\mathfrak{c}$-towers being bases for large ultrafilters on X.

Let $\{R_\alpha\colon \alpha < \mathfrak{c}\}$ be a well ordering of the family of all infinite subsets of X. Note that if a free filterbase $\{F_\alpha\colon \alpha < \mathfrak{c}\}$ is such that for each α one of the sets $F_\alpha \cap R_\alpha$ and $F_\alpha \setminus R_\alpha$ is empty, then it is a base for an ultrafilter.

To construct the required $\mathfrak{c}$-towers, we proceed by induction. We construct families $\{S_\alpha\colon \alpha < \mathfrak{c}\}$ of infinite subsets of X such that:

(1) each member of S_α is large relatively to some subset of N and each two members of S_α are almost disjoint (i.e., they have an empty or finite intersection); for $n \in N$ each element of S_n is contained in the set $X_{n+1} \cup X_{n+2} \cup \ldots$;

(2) if $\beta < \alpha$, then S_α is $<$ -refinement of S_β, i.e., if $T \in S_\alpha$, then there exists a U in S_β such that $T < U$;

(3) for any $\alpha < \mathfrak{c}$, if L is a filterbase consisting of members of the families S_β with $\beta < \alpha$, then there exist two different members T'_L and T''_L of S_α such that $T'_L < T$ and $T''_L < T$ for any member T of L;

(4) for each α, if $T \in S_\alpha$, then one of sets $T \cap R_\alpha$ and $T \setminus R_\alpha$ is empty.

Let $S_0 = \{R_0\}$ for R_0 large relatively to some subset of N, and let $S_0 = \{X \setminus R_0\}$ in the opposite case.

Assume that the families S_β for $\beta < \alpha$, where $\alpha < \mathfrak{c}$, are already defined. Let L be a filterbase contained in $\bigcup \{S_\beta\colon \beta < \alpha\}$ and such that $L \cap S_\beta \neq \emptyset$ for $\beta < \alpha$. Since each element of S_β is large relatively to some subset of N, the filterbase L is large. By (1), the filterbase L has exactly one member in each S_β; in particular, cardinality of L is less than $\mathfrak{c}$. By Lemma 4, there exists a subset T of X which is large relatively to some subset of N and such that $T < F$ for each $F \in L$. Divide T into two sets T' and T'', disjoint and large relatively to some subsets of N. One of the sets $T' \cap R_\alpha$ and $T' \setminus R_\alpha$ is large relatively to some subset of N. Choose that one and denote it by T'_L. In an analogous way choose T''_L. The families $L \cup \{T'_L\}$ and $L \cup \{T''_L\}$ are filterbases on X. Define S_α to be the family of all sets T'_L and T''_L chosen for arbitrary filterbases L contained in $\bigcup \{S_\beta\colon \beta < \alpha\}$ and such that $L \cap S_\beta \neq \emptyset$ for $\beta < \alpha$.

The construction of S_a assures that conditions (1)-(4) are satisfied for $\beta \leqslant a$ if they are satisfied for $\beta < a$. Having the families S_a, $a < \mathfrak{c}$, with properties (1)-(4), we get $2^{\mathfrak{c}}$ selectors each of which is a filterbase. By (4), each of them is a filterbase for an ultrafilter on X. Being a $\mathfrak{c}$-tower, by (2), it is a base for an absolute ultrafilter.

REFERENCES

[1] A. Blass, *The Rudin-Keisler ordering of P-points*, Transactions of the American Mathematical Society 179 (1973), p. 145-166.

[2] D. Booth, *Ultrafilters on a countable set*, Annals of Mathematical Logic 2 (1970-1971), p. 1-24.

[3] E. van Douwen, *MA and pathological points in βX*, preprint.

[4] R. Engelking, *Hausdorff's gaps and limits and compactifications*, p. 89-94 in *Theory of sets and topology* (A collection of papers in honour of Felix Hausdorff), Berlin 1972.

[5] S. H. Hechler, *Short complete nested sequences of $\beta N - N$ and small maximal almost-disjoint families*, General Topology and its Applications 2 (1972), p. 139-149.

[6] A. Kucia and A. Szymański, *Absolute points in $\beta N - N$*, preprint.

[7] K. Kunen and F. D. Tall, *Between Martin's Axiom and Suslin's Hypothesis*, preprint.

[8] F. Rothberger, *On some problems of Hausdorff and Sierpiński*, Fundamenta Mathematicae 35 (1948), p. 29-46.

[9] F. C. Solomon, *Families of sets and functions*, preprint.

[10] A. Szymański, *The existence of $P(m)$-points of N for $\aleph_0 < m < \mathfrak{c}$*, preprint.

[11] F. D. Tall, *Hausdorff's gaps and limits and a new iterated extension axiom*, preprint.

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