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THE REGULAR COMPONENT OF A GROUP-VALUED SET FUNCTION

BY

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1. Introduction, notation and terminology. Throughout G denotes a complete Hausdorff topological Abelian group, $\mathfrak{R}$ stands for a ring of sets, and $\mathfrak{K}$ for a subfamily of $\mathfrak{R}$ closed under finite unions. It is a consequence of a result due to Traynor [5] that every locally exhaustive additive set function $\mu\colon \mathfrak{R} \to G$ can be uniquely decomposed in the form $\mu = \mu_1 + \mu_2$, where $\mu_1, \mu_2\colon \mathfrak{R} \to G$ are locally exhaustive additive set functions, μ_1 is (inner) $\mathfrak{K}$-regular, and μ_2 is $\mathfrak{K}$-antiregular (see the proof of Theorem 2 below). Earlier related results can be found in [2] and [3]; see also [6], (3.6) (a). The aim of our paper is to give explicit formulae for μ_1 and μ_2 (see (5) and (6) below).

For every $\mathfrak{S} \subset \mathfrak{R}$ and $A \in \mathfrak{R}$ we set

$$\mathfrak{S}_A = \{B \in \mathfrak{S} \mid B \subset A\}.$$

For every $A \in \mathfrak{R}$ we denote by $\Delta(A)$ the family of all finite partitions of A contained in $\mathfrak{R}$, and we define an order relation $\preccurlyeq$ on $\Delta(A)$ by setting $\mathfrak{Z} \preccurlyeq \mathfrak{M}$ if for each $B \in \mathfrak{M}$ there exists $C \in \mathfrak{Z}$ with $B \subset C$. Clearly, $\Delta(A)$ is directed by $\preccurlyeq$.

Let $B \in \mathfrak{R}_A$ and $\mathfrak{Z} \in \Delta(A)$. If B is the union of a subfamily of $\mathfrak{Z}$, we write

$$\mathfrak{Z}|_B = \{C \in \mathfrak{Z} \mid C \subset B\}.$$

Throughout $\mathfrak{U}$ denotes the family of all closed symmetric neighbourhoods of 0 in G. For $U \in \mathfrak{U}$ we put

$$U^{(n)} = U + \ldots + U \ (n \text{ summands}).$$

An additive set function $\mu\colon \mathfrak{R} \to G$ is called

locally exhaustive if $\mu(A_n) \to 0$ whenever (A_n) is a disjoint sequence in $\mathfrak{R}_A$ and $A \in \mathfrak{R}$;

σ-additive if

$$\sum_{m=1}^{n} \mu(A_m) \to \mu\Big(\bigcup_{m \in N} A_m\Big)$$

whenever (A_m) is a disjoint sequence in $\mathfrak{R}$ with $\bigcup_{m \in N} A_m \in \mathfrak{R}$;

$\mathfrak{K}$-regular if given $A \in \mathfrak{R}$ and $U \in \mathfrak{U}$ there exists $K \in \mathfrak{K}_A$ such that $\mu(\mathfrak{R}_{A \setminus K}) \subset U$.

We set

$$\mathrm{ea}(\mathfrak{R}; G) = \{\mu\colon \mathfrak{R} \to G \mid \mu \text{ is additive and locally exhaustive}\}.$$

In the sequel, μ always denotes an element of $\mathrm{ea}(\mathfrak{R}; G)$.

A ring topology $\mathfrak{T}$ on $\mathfrak{R}$ is called an *FN-topology* if it admits a base of neighbourhoods of $\emptyset$ consisting of hereditary subfamilies of $\mathfrak{R}$ (see, e.g., [5], 1.3). We say that $\mathfrak{T}$ is *$\mathfrak{K}$-regular* provided for every $A \in \mathfrak{R}$ and every $\mathfrak{T}$-neighbourhood W of $\emptyset$ there exists $K \in \mathfrak{K}_A$ with $\mathfrak{R}_{A \setminus K} \subset W$.

We denote by $\mathfrak{T}_\mu$ the weakest FN-topology on $\mathfrak{R}$ with respect to which μ is continuous. Then $\mathfrak{T}_\mu$ is $\mathfrak{K}$-regular if and only if μ is $\mathfrak{K}$-regular.

Finally, we say that μ is *locally $\mathfrak{T}$-singular*, where $\mathfrak{T}$ is an FN-topology on $\mathfrak{R}$, if, given $A \in \mathfrak{R}$, $U \in \mathfrak{U}$ and a $\mathfrak{T}$-neighbourhood W of $\emptyset$, there exists $B \in \mathfrak{R}_A$ with $\mu(\mathfrak{R}_B) \subset U$ and $A \setminus B \in W$ (cf. [5], 1.4, and [6], pp. 472–473). If $\mathfrak{T} = \mathfrak{T}_\nu$, where $\nu \in \mathrm{ea}(\mathfrak{R}; G)$, then μ is said to be *locally ν-singular*.

2. Results. We start with an essentially known and easy (cf. [1], 1.5.17)

LEMMA 1. *For every $A \in \mathfrak{R}$ the net $\{\mu(K) \mid K \in \mathfrak{K}_A\}$, where the index set is directed upwards by inclusion, satisfies the Cauchy condition.*

Using Lemma 1, we can define for every $A \in \mathfrak{R}$

$$\psi_\mu(A) = \lim \{\mu(K) \mid K \in \mathfrak{K}_A\}.$$

Moreover, we put for every $\mathfrak{Z} \in \Delta(A)$

$$\phi_\mu(\mathfrak{Z}) = \sum_{Z \in \mathfrak{Z}} \psi_\mu(Z).$$

LEMMA 2. *Let $A \in \mathfrak{R}$, let $\mathfrak{Z} = \{Z_1, \ldots, Z_n\} \in \Delta(A)$ and let $U \in \mathfrak{U}$. Then there exist $K_i \in \mathfrak{K}_{Z_i}$, $i = 1, \ldots, n$, such that for every $\mathfrak{M} = \{M_1, \ldots, M_m\} \in \Delta(A)$ with $\mathfrak{M} \succcurlyeq \mathfrak{Z}$ we have*

$$\mu\Big(\bigcup_{j=1}^{m} L_j \setminus \bigcup_{i=1}^{n} K_i\Big) \in U \quad \textit{whenever } L_j \in \mathfrak{K}_{M_j},\ j = 1, \ldots, m.$$

Moreover, $\phi_\mu(\mathfrak{S}) \in U$ whenever $\mathfrak{S} \in \Delta(D)$ and

$$\mathfrak{S} \succcurlyeq \{(Z_1 \setminus K_1) \cap D, \ldots, (Z_n \setminus K_n) \cap D\} \quad \textit{and} \quad D \subset \bigcup_{i=1}^{n} Z_i \setminus K_i, \quad D \in \mathfrak{R}.$$

Proof. Choose $V \in \mathfrak{U}$ with $V^{(n)} \subset U$ and $K_i \in \mathfrak{K}_{Z_i}$ with

$$\mu(L) - \mu(K_i) \in V \quad \text{whenever } K_i \subset L \in \mathfrak{K}_{Z_i},\ i = 1, \ldots, n$$

(see Lemma 1). Put $N(i) = \{1 \leqslant j \leqslant m \mid M_j \cap Z_i \neq \emptyset\}$. We have

$$Z_i = \bigcup_{j \in N(i)} M_j$$

and

$$\mu\Big(\bigcup_{j\in N(i)} L_j \setminus K_i\Big) = \mu\Big(\Big(\bigcup_{j\in N(i)} L_j \cup K_i\Big)\setminus K_i\Big) \in V.$$

This yields the first part of the assertion. The second part is an easy consequence of the first one.

LEMMA 3. *For every $A \in \mathfrak{R}$ the net $\{\phi_\mu(\mathfrak{Z}) \mid \mathfrak{Z} \in \Delta(A)\}$ satisfies the Cauchy condition.*

Proof. We assume the contrary. Then there exists $V \in \mathfrak{U}$ such that for every $\mathfrak{Z} \in \Delta(A)$ we can find $\mathfrak{M} \in \Delta(A)$ with $\mathfrak{M} \succcurlyeq \mathfrak{Z}$ and

$$(*) \qquad \phi_\mu(\mathfrak{M}) - \phi_\mu(\mathfrak{Z}) \notin V.$$

Choose $V_0 \in \mathfrak{U}$ with $V_0^{(3)} \subset V$ and $V_n \in \mathfrak{U}$ with $\sum_{k=1}^{n} V_k \subset V_0$. We shall construct, by recursion, $K_n \in \mathfrak{K}_A$ and $\mathfrak{Z}_n \in \Delta(A \setminus K_n)$ such that for every $n \in N$ the following three conditions hold:

(i) $K_n \supset K_{n+1}$;

(ii) $\mu(K_n \setminus K_{n+1}) \notin V_0$;

(iii) $\phi_\mu(\mathfrak{M}) - \phi_\mu(\mathfrak{Z}_n) \in \sum_{k=1}^{n} V_k$ whenever $\mathfrak{M} \in \Delta(A \setminus K_n)$ and $\mathfrak{M} \succcurlyeq \mathfrak{Z}_n$.

Clearly, (i) and (ii) contradict the local exhaustivity of μ as $K_n \subset A$.

By Lemma 1, choose $K_1 \in \mathfrak{K}_A$ with

$$\mu(\mathfrak{K}_{A\setminus K_1}) - \mu(\mathfrak{K}_{A\setminus K_1}) \subset V_1$$

and $\mathfrak{Z}_1 \in \Delta(A \setminus K_1)$ arbitrarily.

Suppose now the construction has been carried out until some $n \in N$. Applying $(*)$ to $\mathfrak{Z}_n \cup \{K_n\}$, we obtain $\mathfrak{Z} \in \Delta(A)$ with the following properties:

$$\mathfrak{Z} \succcurlyeq \mathfrak{Z}_n \cup \{K_n\} \quad \text{and} \quad \phi_\mu(\mathfrak{Z}) - (\phi_\mu(\mathfrak{Z}_n) + \mu(K_n)) \notin V.$$

We have $\mathfrak{Z}|_{K_n} = \{B_1, \ldots, B_m\}$. Choose $W \in \mathfrak{U}$ with $W + W \subset V_{n+1}$. By Lemma 2, applied to $\mathfrak{Z}|_{K_n}$ and W, there exist $L_i \in \mathfrak{K}_{B_i}$, $i = 1, \ldots, m$, such that

$$(1) \qquad \sum_{i=1}^{m} \psi_\mu(B_i) - \sum_{i=1}^{m} \mu(L_i) \in W,$$

$$(2) \qquad \phi_\mu(\mathfrak{S}) \in W \quad \text{whenever } \mathfrak{S} \succcurlyeq \{B_1 \setminus L_1, \ldots, B_m \setminus L_m\}.$$

Put

$$K_{n+1} = \bigcup_{i=1}^{m} L_i .$$

and

$$\mathfrak{Z}_{n+1} = \mathfrak{Z}_n \cup \{B_1 \setminus L_1, \ldots, B_m \setminus L_m\}.$$

Clearly, $\mathfrak{Z}_{n+1} \in \Delta(A \setminus K_{n+1})$ and (i) holds. By the definition of B_i's and K_{n+1}, we have

$$\phi_\mu(\mathfrak{Z}) = \phi_\mu(\mathfrak{Z}|_{A\setminus K_n}) + \sum_{i=1}^{m} \psi_\mu(B_i),$$

whence, in view of (1),

$$\phi_\mu(\mathfrak{Z}) - \phi_\mu(\mathfrak{Z}|_{A\setminus K_n}) - \mu(K_{n+1}) \in V_0.$$

By (iii) and our choice of $\mathfrak{Z}$, we have

$$\phi_\mu(\mathfrak{Z}|_{A\setminus K_n}) - \phi_\mu(\mathfrak{Z}_n) \in V_0.$$

It follows that

$$\phi_\mu(\mathfrak{Z}) - (\phi_\mu(\mathfrak{Z}_n) + \mu(K_{n+1})) \in V_0 + V_0,$$

which yields, in view of our choice of $\mathfrak{Z}$, (ii) for $n+1$.

In order to check (iii), fix $\mathfrak{M} \in \Delta(A \setminus K_{n+1})$ with $\mathfrak{M} \succcurlyeq \mathfrak{Z}_{n+1}$. We have $\mathfrak{M}|_{A\setminus K_n} \succcurlyeq \mathfrak{Z}_n$, so that, by the inductive hypothesis,

$$\phi_\mu(\mathfrak{M}|_{A\setminus K_n}) - \phi_\mu(\mathfrak{Z}_n) \in \sum_{k=1}^{n} V_k. \tag{3}$$

Since

$$K_n \setminus K_{n+1} = \bigcup_{i=1}^{m} (B_i \setminus L_i),$$

we get, in view of (2) and $\mathfrak{M}|_{K_n\setminus K_{n+1}} \succcurlyeq \mathfrak{Z}_{n+1}|_{K_n\setminus K_{n+1}}$,

$$\phi_\mu(\mathfrak{M}|_{K_n\setminus K_{n+1}}) - \phi_\mu(\mathfrak{Z}_{n+1}|_{K_n\setminus K_{n+1}}) \in V_{n+1}. \tag{4}$$

Summing up (3) and (4), we get (iii) for $n+1$.

Using Lemma 3, we can define set functions

$$\mu_r, \mu_{ar}\colon \mathfrak{R} \to G$$

by the formulae

$$\mu_r(A) = \lim \{\phi_\mu(\mathfrak{Z}) |\ \mathfrak{Z} \in \Delta(A)\}, \tag{5}$$

$$\mu_{ar}(A) = \mu(A) - \mu_r(A) \tag{6}$$

for every $A \in \mathfrak{R}$.

LEMMA 4. *Let $A \in \mathfrak{R}$ and $V \in \mathfrak{U}$. Then there exists $K \in \mathfrak{K}_A$ such that*

$$\mu_r(A) - \mu(K) \in V \quad \textit{and} \quad \mu_r(\mathfrak{R}_{A\setminus K}) \subset V.$$

Proof. Choosing $U \in \mathfrak{U}$ with $U + U \subset V$, we find $\mathfrak{Z} \in \Delta(A)$, $\mathfrak{Z} = \{Z_1, \ldots, Z_n\}$, satisfying $\mu_r(A) - \phi_\mu(\mathfrak{Z}) \in U$. Let $K_i \in \mathfrak{K}_{Z_i}$ be given by Lemma 2. Put

$$K = \bigcup_{i=1}^{n} K_i.$$

Then, in view of that lemma, $\phi_\mu(\mathfrak{Z})-\mu(K)\in U$. It follows that $\mu_r(A)-\mu(K)\in V$.

Fix $D\in\mathfrak{R}_{A\setminus K}$ and let $\mathfrak{M}\in\Delta(D)$ satisfy

$$\mathfrak{M}\succcurlyeq\{(Z_1\setminus K_1)\cap D,\ \ldots,(Z_n\setminus K_n)\cap D\}.$$

Then, in view of Lemma 2, $\phi_\mu(\mathfrak{M})\in U$, so that $\mu_r(D)\in U$.

THEOREM 1. *For every* $\mu\in\mathrm{ea}(\mathfrak{R};G)$ *we have* $\mu_r,\ \mu_{ar}\in\mathrm{ea}(\mathfrak{R};G)$, *and* μ_r *is* $\mathfrak{K}$*-regular. Moreover,* μ *is* $\mathfrak{K}$*-regular if and only if* $\mu=\mu_r$.

Proof. We first show that μ is additive. Let $A,B\in\mathfrak{R}$ be disjoint. Fix $V\in\mathfrak{U}$ and choose $U\in\mathfrak{U}$ with $U^{(3)}\subset V$. There exists $\mathfrak{Z}_0\in\Delta(A\cup B)$ such that

$$\mu_r(A\cup B)-\phi_\mu(\mathfrak{Z})\in U\quad\text{ whenever } \mathfrak{Z}\in\Delta(A\cup B)\text{ and }\mathfrak{Z}\succcurlyeq\mathfrak{Z}_0.$$

Fix $\mathfrak{M}_0\in\Delta(A)$ and $\mathfrak{N}_0\in\Delta(B)$ with $\mathfrak{M}_0\cup\mathfrak{N}_0\succcurlyeq\mathfrak{Z}_0$ and

$$\mu_r(A)-\phi_\mu(\mathfrak{M}_0)\in U\quad\text{ and }\quad\mu_r(B)-\phi_\mu(\mathfrak{N}_0)\in U.$$

It follows that

$$\mu_r(A\cup B)-\big(\mu_r(A)+\mu_r(B)\big)\in U^{(3)}\subset V.$$

Since V is arbitrary, this yields $\mu_r(A\cup B)=\mu_r(A)+\mu_r(B)$.

As μ is locally exhaustive, it follows easily from Lemma 4 that μ_r is also locally exhaustive. The $\mathfrak{K}$-regularity of μ_r follows also immediately from Lemma 4.

Finally, suppose μ is $\mathfrak{K}$-regular. Then $\psi_\mu(A)=\mu(A)$ for every $A\in\mathfrak{R}$. Hence $\phi_\mu(\mathfrak{Z})=\mu(A)$ whenever $A\in\mathfrak{R}$ and $\mathfrak{Z}\in\Delta(A)$. This yields $\mu_r(A)=\mu(A)$ for every $A\in\mathfrak{R}$.

Remarks. 1. If $\mathfrak{R}$ is a δ-ring of sets and $\mu\in\mathrm{ea}(\mathfrak{R};G)$ is σ-additive, then, in the definitions of ϕ_μ and μ_r, one can replace $\Delta(A)$ by the family of all countable partitions of A contained in $\mathfrak{R}$. Moreover, μ_r and μ_{ar} are σ-additive.

2. The second assertion of Lemma 4 states, in the terminology of [4], 1.3 (local setting), that μ_r is locally nearly supported on $\mathfrak{K}$. Moreover, if $\mathfrak{K}$ is an ideal of $\mathfrak{R}$, then $\mu_{ar}(\mathfrak{K})=\{0\}$. Thus, the decomposition considered in this paper is then a local version of Traynor's decomposition ([4], Theorem 1.7).

The next three lemmas will serve us to prove that μ_r and μ_{ar} are, indeed, the $\mathfrak{K}$-regular and the $\mathfrak{K}$-antiregular components of μ in the sense of Traynor [5].

LEMMA 5. *If* $\mu,\nu\in\mathrm{ea}(\mathfrak{R};G)$, *then*

(a) $\psi_{(\mu+\nu)}=\psi_\mu+\psi_\nu$;

(b) $(\mu+\nu)_r=\mu_r+\nu_r$, $(\mu+\nu)_{ar}=\mu_{ar}+\nu_{ar}$.

LEMMA 6. *Suppose* $\mu_1,\mu_2\in\mathrm{ea}(\mathfrak{R};G)$ *and* $\mu=\mu_1+\mu_2$. *If* $(\mu_1)_r=\mu_1$ *and* $(\mu_2)_r=0$, *then* $\mu_1=\mu_r$ *and* $\mu_2=\mu_{ar}$.

Proof. By Lemma 5 (b),

$$\mu_r = (\mu_1 + \mu_2)_r = (\mu_1)_r + (\mu_2)_r = \mu_1 .$$

It is worth-while to note the following assertion, even though it will not be used in the sequel.

PROPOSITION. *If* $\mu \in \mathrm{ea}(\mathfrak{R}; G)$, *then*

(a) $(\mu_r)_r = \mu_r$, $(\mu_{ar})_{ar} = \mu_{ar}$;

(b) $(\mu_r)_{ar} = 0 = (\mu_{ar})_r$.

Proof. The first part of (a) follows from Theorem 1. The remaining assertions can be deduced from this and Lemma 5 (b).

LEMMA 7. *If* μ *is locally* μ_r*-singular, then* $\mu_r = 0$.

Proof. Let $A \in \mathfrak{R}$ and $V \in \mathfrak{U}$. Choose $U \in \mathfrak{U}$ with $U + U \subset V$. By assumption, there exists $B \in \mathfrak{R}_A$ such that

$$\mu(\mathfrak{R}_B) \subset U \quad \text{and} \quad \mu_r(\mathfrak{R}_{A \setminus B}) \subset U .$$

Then, obviously, $\mu_r(\mathfrak{R}_B) \subset U$, whence $\mu_r(A) \in V$. Since V is arbitrary, we conclude that $\mu_r(A) = 0$.

THEOREM 2. *Let* $\mu \in \mathrm{ea}(\mathfrak{R}; G)$ *and let* $\mathfrak{T}_0$ *be the strongest* $\mathfrak{K}$*-regular FN-topology on* $\mathfrak{R}$. *Then* μ_r *is locally* $\mathfrak{T}_0$*-continuous and* μ_{ar} *is locally* $\mathfrak{T}_0$*-singular.*

Proof. That $\mathfrak{T}_0$ exists follows from the simple observation that the family of $\mathfrak{K}$-regular FN-topologies on $\mathfrak{R}$ is closed under arbitrary suprema.

By Traynor's decomposition theorem ([5], 6.3), there exist (uniquely determined) $\mu_1, \mu_2 \in \mathrm{ea}(\mathfrak{R}; G)$ such that μ_1 is locally $\mathfrak{T}_0$-continuous, μ_2 is locally $\mathfrak{T}_0$-singular, and $\mu = \mu_1 + \mu_2$. Thus, it is enough to show that $\mu_1 = \mu_r$ or, in view of Lemma 6, that $(\mu_1)_r = \mu_1$ and $(\mu_2)_r = 0$. The first assertion follows directly from Theorem 1 as μ_1 is $\mathfrak{K}$-regular. The same theorem yields that $(\mu_2)_r$ is $\mathfrak{K}$-regular, so that μ_2 is locally $(\mu_2)_r$-singular. Accordingly, the second assertion is a consequence of Lemma 7.

Remark. 3. In the case where $G = \boldsymbol{R}$, the additive group of the reals with the usual topology, the decomposition considered in this paper coincides with the Riesz decomposition in the Dedekind complete Riesz space $\mathrm{ea}(\mathfrak{R}; \boldsymbol{R})$ with respect to the band of $\mathfrak{K}$-regular elements of $\mathrm{ea}(\mathfrak{R}; \boldsymbol{R})$. Indeed, $\mu, \nu \in \mathrm{ea}(\mathfrak{R}; \boldsymbol{R})$ are orthogonal if and only if μ is locally ν-singular.

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