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QUASI-CONSTANTS IN GENERAL ALGEBRAS

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In this paper we shall use the terminology and notation of [2] and [3]. For subalgebras $\mathfrak{B} = (B; \boldsymbol{F}|_B)$ of a fixed algebra $\mathfrak{A} = (A; \boldsymbol{F})$ $(B \subset A)$ we shall often write B instead of $\mathfrak{B}$ (e.g., by a homomorphism $h: B \to A$ we shall mean the homomorphism of the subalgebra $\mathfrak{B}$ into the algebra $\mathfrak{A}$).

Recently, Professor E. Marczewski proposed a notion of independence of subalgebras of a general (abstract) algebra. Two subalgebras B_1 and B_2 of $\mathfrak{A} = (A; \boldsymbol{F})$ are *independent* if, for every pair of homomorphisms $h_i: B_i \to A$ $(i = 1, 2)$, there exists a homomorphism $h: C(B_1 \cup B_2) \to A$ such that $h|_{B_i} = h_i$ $(i = 1, 2)$. In such a case $h_1(a) = h_2(a) = a$ for any $a \in B_1 \cap B_2$. Consequently, the following property of elements of an algebra seems to be interesting for the study of algebras:

An element a of the subalgebra B of $\mathfrak{A}$ is a *quasi-constant with respect to B* if $h(a) = a$ for any homomorphism $h: B \to A$.

1. We denote by $Q(B)$ the set of all quasi-constants with respect to B in $\mathfrak{A}$. It is easy to see that

(i) *If $Q(B) \neq \emptyset$, then $Q(B)$ is a subalgebra of B.*

(ii) *$Q(B_1) \subset Q(B_2)$ for subalgebras $B_1 \subset B_2 \subset A$.*

(iii) *Every element of $Q(B)$ is self-dependent* (self-dependent elements were examined in [4]).

(iv) *$Q(B) \subset h(B)$ for each homomorphism $h: B \to A$.*

PROPOSITION 1. *If a is the only fixed point of a unary algebraic operation f of $\mathfrak{A}$ (i.e., $f(a) = a$ and $f(b) \neq b$ for $b \neq a$), then $a \in Q(C(\{a\}))$.*

In fact, let h be a homomorphism of $C(\{a\})$ into A. Then

$$f(h(a)) = h(f(a)) = h(a).$$

Hence $h(a)$ is a fixed point of f and, consequently, $h(a) = a$.

The following two statements are easy consequences of this proposition:

(v) $c \in Q(C(\{c\}))$ *for every* $c \in C(\emptyset)$.

(vi) $C(\emptyset) = Q(C(\emptyset)) \subset Q(B)$ *for every subalgebra* B *of* $\mathfrak{A}$.

Note that, in general, the inclusion in (vi) is proper, as it is shown by the following example: $\mathfrak{A} = (\{a, b, c\}; f)$, where $f(a) = b$, $f(b) = a$, $f(c) = c$. Here $C(\emptyset) = \emptyset$ and, by virtue of Proposition 1, c is a quasi-constant.

It seems worth to note that from (iv) it follows $Q(A) \subset \{c\}$ if $\{c\}$ is a subalgebra of $\mathfrak{A}$. Hence, if an algebra has exactly one one-element subalgebra $\{c\}$ (e.g., an algebraic constant), then $Q(A) = \{c\}$. Consequently, $Q(A) = \{0\} = C(\emptyset)$ for groups (abelian or not), rings, modules, and, in general, for Ω-groups (for the definition see [1], p. 115). It is also clear that $Q(A) = \emptyset = C(\emptyset)$ whenever $\mathfrak{A}$ has at least two idempotent elements, which is valid in idempotent (at least two-element) algebras, e.g., in lattices and in diagonal algebras.

Another class of algebras with at most one quasi-constant can be obtained from

PROPOSITION 2. *If in an algebra* $\mathfrak{A} = (A; F)$ *each unary algebraic operation is an endomorphism, then there is at most one quasi-constant (with respect to* A*) in* $\mathfrak{A}$.

Indeed, if $c \in Q(A)$, then $f(c) = c$ for every unary algebraic operation f. Hence $Q(A) \subset \{c\}$, since $\{c\}$ is a one-element subalgebra of $\mathfrak{A}$.

Note that the above-given assumption is satisfied in any commutative algebra. Recall that $\mathfrak{A} = (A; F)$ is *commutative* if

$$f(g(x_{11}, \ldots, x_{1n}), \ldots, g(x_{m1}, \ldots, x_{mn})) = g(f(x_{11}, \ldots, x_{m1}), \ldots, f(x_{1n}, \ldots, x_{mn}))$$

for any algebraic operations f (m-ary) and g (n-ary). In particular, in groupoids in which $(xy)(uv) = (xu)(yv)$ (moreover, in groupoids with $(xy)^2 = x^2y^2$), and thus in medial semigroups, the set $Q(A)$ is at most one-element and $c \in Q(A)$ is the only idempotent element. This is also true in generalized diagonal algebras (for the definition see [5]).

From the following theorem we obtain a class of algebras in which $Q(A) = C(\emptyset)$:

THEOREM 1. *Let* I *be an independent set in an algebra* $\mathfrak{A}$ *and let* $c \in C(I)$. *Then* c *is a quasi-constant with respect to* $C(I)$ *if and only if* c *is an algebraic constant in* $\mathfrak{A}$.

Proof. Let $c = f(a_1, \ldots, a_n) \in Q(C(I))$, where f is an n-ary algebraic operation and $a_1, \ldots, a_n \in I$. There is a homomorphism $h_0\colon C(I) \to A$ such that $h_0(a_1) = \ldots = h_0(a_n) = a_1$. Define a unary algebraic operation g by $g(x) = f(x, \ldots, x)$. We have

$$g(a_1) = f(h_0(a_1), \ldots, h_0(a_n)) = h_0(c) = c.$$

Observe that, for each $x \in A$, there exists a homomorphism $h\colon C(I) \to A$ such that $h(a_1) = x$. Hence $g(x) = g(h(a_1)) = h(g(a_1)) = h(c) = c$ and $c \in C(\emptyset)$ which completes the proof.

COROLLARY. *If $\mathfrak{A}$ has a basis, then $Q(A) = C(\emptyset)$.*

It would be interesting to know the answer to the following

PROBLEM 1. For which algebras $Q(A) = C(\emptyset)$? (**P 877**)

PROBLEM 2. For which algebras $Q(A) = \emptyset$? (**P 878**)

2. The following example shows that, in general, $Q(Q(B)) \neq Q(B)$. Let $\mathfrak{A} = (A; f, g)$, where $A = \{a, b, c, d, e\}$ and

$$f(x) = \begin{cases} b & \text{for } x = a, \\ c & \text{for } x = b, \\ a & \text{for } x = c, d, e, \end{cases} \qquad g(x) = \begin{cases} c & \text{for } x = a, \\ b & \text{for } x = c, \\ a & \text{for } x = b, d, e. \end{cases}$$

Put $B = A$. Then $Q(B) = \{a, b, c\}$ and $Q(Q(B)) = \emptyset$.

Put $Q^0(B) = B$ and $Q^n(B) = Q(Q^{n-1}(B))$ for $n = 1, 2, \ldots$ Observe that if $Q^k(B) = Q^l(B)$ for some $l > k$, then $Q^k(B) = Q^{k+1}(B) = \ldots$ If $Q^n(A)$ is a one-element set for some $n \geqslant 1$, then we have

$$Q^1(A) = \ldots = Q^n(A) = Q^{n+1}(A) = \ldots$$

Other possible sequences $Q^n(B)$ are given by the following theorem which answers a question formulated by Jerzy Płonka:

THEOREM 2. *Let n be a natural number and let $\varkappa \neq 1$ be a cardinal number. Then the three cases can be realized in unary algebras:*

(1) $Q(B) \supset Q^2(B) \supset \ldots \supset Q^n(B) = Q^{n+1}(B)$, $|Q^n(B)| = \varkappa$;

(2) $Q(B) \supset Q^2(B) \supset \ldots$, $\left|\bigcap_{k=1}^{\infty} Q^k(B)\right| = \varkappa$;

(3) $Q(B) = Q^2(B) = \ldots$, $|Q(B)| = \varkappa$ *or* 1;

where the inclusions in (1) *and* (2) *are proper.*

First, we prove the following

LEMMA. *Let $\mathfrak{A}_0 = (A_0; \boldsymbol{F}_0)$ be a unary algebra without one-element subalgebras. Then there exists a unary algebra $\mathfrak{A}_1 = (A_1; \boldsymbol{F}_1)$ such that A_0 is a proper subalgebra of $\mathfrak{A}_1$ and $Q(A_1) = A_0$.*

Proof. Let $A_1 = A_0 \cup \{b, c\}$, where $b \neq c$ and $b, c \notin A_0$. We define, for all $f_0 \in \boldsymbol{F}_0$, the operation

$$f_1(x) = \begin{cases} f_0(x) & \text{for } x \in A_0, \\ x & \text{for } x = b, c. \end{cases}$$

Put, for any $a \in A_0$,

$$g_{1,a}(x) = \begin{cases} a & \text{for } x = b, c, \\ x & \text{for } x \in A_0. \end{cases}$$

Write

$$\boldsymbol{F}_1 = \{f_1 \colon f_0 \in \boldsymbol{F}_0\} \cup \{g_{1,a} \colon a \in A_0\}.$$

Let $h\colon A_1 \to A_1$ be an endomorphism of $\mathfrak{A}_1$. If $h(b) \in A_0$, then, for any $a \in A_0$, we have

$$h(a) = h\big(g_{1,a}(b)\big) = g_{1,a}\big(h(b)\big) = h(b).$$

Thus $\{h(b)\} = h(A_0)$ is a one-element subalgebra of $\mathfrak{A}_0$, which is a contradiction. Hence $h(b) \in \{b, c\}$ and we have

$$h(a) = h\big(g_{1,a}(b)\big) = g_{1,a}\big(h(b)\big) = a \quad \text{for each } a \in A_0,$$

i.e., $A_0 \subset Q(A_1)$. Put

$$h_1(x) = \begin{cases} x & \text{for } x \in A_0, \\ b & \text{for } x = c, \\ c & \text{for } x = b. \end{cases}$$

Obviously, h_1 is an endomorphism of $\mathfrak{A}_1$, which implies $Q(A_1) = A_0$.

Proof of Theorem 2. Iterating the procedure of the lemma, we get a sequence of algebras $\mathfrak{A}_k = (A_k; \boldsymbol{F}_k)$ which have no one-element subalgebras $(k = 0, 1, \ldots)$ and $Q(A_k) = A_{k-1}$ in $\mathfrak{A}_k$ for $k \geqslant 1$. Now, let us define an algebra $\mathfrak{A} = (A; \boldsymbol{F})$ by putting

$$A = \bigcup_{k=0}^{\infty} A_k \quad \text{and} \quad \boldsymbol{F} = \bigcup_{k=1}^{\infty} \{\bar{g}_k \colon g_k \in \boldsymbol{F}_k\},$$

where

$$\bar{g}_k(x) = \begin{cases} g_k(x) & \text{for } x \in A_k, \\ x & \text{in the other case.} \end{cases}$$

Note that $h(A_k) = A_k$ for each homomorphism $h\colon A_k \to A_k$, $k = 0, 1, \ldots$ Taking $|A_0| = \varkappa$ and $C(\emptyset) = A_0$ in the algebra $\mathfrak{A}_0$ and $B = A_n$, we get case (1) for $\varkappa > 1$. For $\varkappa = 0$, we get (1) by considering the algebra $\mathfrak{A}_1 = (\{b, c\}; g)$, where $g(b) = c$ and $g(c) = b$, and similar constructions of $\mathfrak{A}_k$ for $k = 2, 3, \ldots$

To prove case (2) let us assume that $B_0, B_1, \ldots$ are pairwise disjoint sets such that $|B_0| = \varkappa$, $|B_1| < |B_2| < \ldots$ and $|B_k|$ are prime natural numbers for $k \geqslant 1$. Put

$$A = \bigcup_{k=0}^{\infty} B_k$$

and define unary operations f_a, g_b, c_b and p on A as follows:

$$f_a(x) = \begin{cases} a & \text{for } x \in B_k, \\ x & \text{in the other case} \end{cases}$$

for every $a \in B_{k+1}$, $k = 1, 2, \ldots$,

$$g_b(x) = \begin{cases} x & \text{for } x \in B_0, \\ b & \text{in the other case,} \end{cases}$$

$$c_b(x) = b$$

for each $b \in B_0$, and

$$p(x) = \begin{cases} x & \text{for } x \in B_0, \\ p_k(x) & \text{for } x \in B_k \quad (k = 1, 2, \ldots), \end{cases}$$

where p_k is a cyclic permutation of B_k.

Let now

$$F = \{f_a: a \in \bigcup_{k=2}^{\infty} B_k\} \cup \{g_b: b \in B_0\} \cup \{c_b: b \in B_0\} \cup \{p\}$$

and $\mathfrak{A} = (A; F)$. We claim that, for every homomorphism $h: C(B_k) \to A$, we have $h(B_k) \subset B_k$ for $k = 0, 1, \ldots$ If $k = 0$, this is obvious. Suppose that $h(x) \in B_0$ for some $x \in B_k$, $k > 0$. Then there exists a $b \in B_0 \setminus \{h(x)\}$. Therefore,

$$h(x) = g_b(h(x)) = h(g_b(x)) = h(b) = b \neq h(x)$$

which is a contradiction. Hence

$$h(x) \in \bigcup_{k=1}^{\infty} B_k \quad \text{for every } x \in B_k \ (k > 0).$$

Taking into account the definition of the operation p, we get $h(x) \in B_k$. We prove by induction that

$$Q^k(A) = B_0 \cup \bigcup_{j=k+1}^{\infty} B_j.$$

This is clear for $k = 0$. Suppose that this is also true for some $k > 0$. Since $B_k \cap Q^k(A) = \emptyset$,

$$h(x) = \begin{cases} p(x) & \text{for } x \in B_{k+1}, \\ x & \text{in the other case} \end{cases}$$

is a homomorphism of $Q^k(A)$ into A. Hence, no element of B_{k+1} is a quasi-constant with respect to $Q^k(A)$. If $a \in B_{k+i}$ for some $i \geqslant 2$, then a is a quasi-constant with respect to $Q^k(A)$, since $h(a) = h(f_a(x)) = f_a(h(x)) = a$ for every homomorphism $h: Q^k(A) \to A$ and for $x \in B_{k+i-1}$. In virtue of (vi), we have $B_0 \subset Q^{k+1}(A)$, and so we obtain

$$Q^{k+1}(A) = B_0 \cup \bigcup_{j=k+2}^{\infty} B_j.$$

Case (2) is proved by taking $B = A$.

For case (3) let us take an algebra $\mathfrak{A}$ such that $C(\emptyset) = B$ and $|B| = \varkappa$ or 1.

3. In the set $Q(A)$ of an algebra $\mathfrak{A} = (A; \boldsymbol{F})$ we distinguish elements a such that $a \in Q(C(\{a\}))$. These elements are called *semi-constants*. By (v), any constant is a semi-constant. Note that Proposition 1 is concerned with semi-constants.

In general, semi-constants do not form a subalgebra. For example, $\mathfrak{A} = (\{a, b, c\}; f, g)$, where $f(a) = g(b) = b$, $f(b) = g(a) = g(c) = a$ and $f(c) = c$. Here $a = g(c) \notin Q(C(\{a\}))$, while $c \in Q(C(\{c\}))$. Nevertheless, the following statement is true:

PROPOSITION 3. *Let $a_1, \ldots, a_n$ be semi-constants in the algebra $\mathfrak{A}$ and let $b \in C(\{a_1, \ldots, a_n\})$. Then*

(α) *b is a quasi-constant with respect to $C(\{a_1, \ldots, a_n\})$;*

(β) *if $C(\{b\}) = C(\{a_1, \ldots, a_n\})$, then b is a semi-constant.*

In fact, if $b = f(a_1, \ldots, a_n)$ for some algebraic operation f, then

$$h(b) = h(f(a_1, \ldots, a_n)) = f(h(a_1), \ldots, h(a_n)) = b$$

for each homomorphism $h\colon C(\{a_1, \ldots, a_n\}) \to A$. Note that ($\beta$) follows easily from ($\alpha$), because

$$b \in Q(C(\{a_1, \ldots, a_n\})) = Q(C(\{b\})).$$

It seems worth to remark that from $C(\{b\}) \supset C(\{a_1, \ldots, a_n\})$, where $a_1, \ldots, a_n$ are semi-constants, $b \in Q(C(\{b\}))$ does not follow (e.g., $\mathfrak{A} = (\{a, b\}; f)$, where $f(a) = f(b) = a$).

Another example shows that, in general, $C(B) \neq Q(A)$, where B is the set of all semi-constants of $\mathfrak{A}$. Put $\mathfrak{A} = (\{a, b, c, d\}; f, g)$, where $f(a) = g(a) = f(d) = g(d) = b$, $f(b) = g(b) = f(c) = a$ and $g(c) = c$. Here $Q(A) = A \neq Q(C(\{c\})) = \{a, b, c\}$.

PROBLEM 3. In which algebras the set of all semi-constants forms a subalgebra? (**P 879**)

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