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CONVERGENCE IN THE DUAL OF CERTAIN $K\{M_p\}$-SPACES

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In this note we give two theorems characterizing convergent sequences in the dual of certain $K\{M_p\}$-spaces (see [3], Chapter II for material on $K\{M_p\}$-spaces). Theorem 6 gives a characterization in terms of the usual representation of elements in $K\{M_p\}'$, when $\{M_p\}$ satisfies conditions (M), (N) and (P) (see [3], II. 2.3 and 4.2); this result is somewhat analogous to the convergence criteria for $\mathscr{S}'$ given in Theorem 56, Chapter 3 of [2], and to the convergence criteria for H'_r given in Theorem 3 of [8]. Theorem 7 gives a characterization in terms of regularizers. There do not seem to be any results analogous to Theorem 7 recorded, even for the case where $K\{M_p\} = \mathscr{S}$.

First, we recall some facts pertinent to $K\{M_p\}$-spaces. Let $\{M_p\}$ be a sequence of extended real-valued functions defined on $\boldsymbol{R}^m$ such that $1 \leqslant M_1(x) \leqslant M_2(x) \leqslant \ldots$ It is further assumed that at each point $x \in \boldsymbol{R}^m$ all the $M_p(x)$ are finite or infinite. If $S = \{x\colon M_p(x) < \infty,\ p \geqslant 1\}$, it is assumed that M_p restricted to S is continuous. An infinitely differentiable function φ defined on $\boldsymbol{R}^m$ belongs to $K\{M_p\}$ if

(1) $D^\alpha \varphi(x) = 0$ for $x \notin S$ and any multi-index α,

(2) $M_p D^\alpha \varphi$ is a continuous bounded function on S for $1 \leqslant p < \infty$ and $0 \leqslant |\alpha| \leqslant p$.

The vector space $K\{M_p\}$ is then given the locally convex topology generated by the norms

$$\|\varphi\|_p = \sup\{M_p(x)|D^\alpha \varphi(x)|\colon x \in S,\ |\alpha| \leqslant p\} \qquad (1 \leqslant p < \infty). \tag{3}$$

We will only consider $K\{M_p\}$-spaces which satisfy three further conditions. The sequence $\{M_p\}$ satisfies the following conditions:

(M) Each M_p is quasi-monotonic, i.e., for $|x'_j| \geqslant |x''_j|$ and x'_j and x''_j having the same sign,

$$M_p(x_1, \ldots, x'_j, \ldots, x_m) \geqslant C_p M_p(x_1, \ldots, x''_j, \ldots, x_m).$$

(N) For each p, there exists an integer $p' > p$ such that the quotient $M_p(x)/M_{p'}(x) = m_{pp'}(x)$ is summable on $\boldsymbol{R}^m$ and $m_{pp'}(x) \to 0$ as $|x| \to \infty$. (Here it is understood that the quotient ∞/∞ is 0.)

(P) For $\varepsilon > 0$ and p an integer, there exist $p' > p$ and N such that $M_p(x) < \varepsilon M_{p'}(x)$ if $|x| > N$ or $M_p(x) > N$.

Many of the familiar test spaces are $K\{M_p\}$-spaces which satisfy conditions (M), (N) and (P).

Example 1. For $K \subseteq \boldsymbol{R}^m$ compact, set $M_p(x) = 1$ if $x \in K$ and $M_p(x) = \infty$ if $x \notin K$. Then $K\{M_p\} = \mathscr{D}_K$ (see [5]), and $\{M_p\}$ satisfies (M), (N) and (P).

Example 2. If $M_p(x) = (1+|x|)^p$, then $K\{M_p\} = \mathscr{S}$ is the space of rapidly decreasing functions [5]. $\{M_p\}$ is easily seen to satisfy conditions (M), (N) and (P).

Example 3. If $M_p(x) = \exp\big(p\gamma(x)\big)$, where $\gamma(x) = (1+|x|^2)^{1/2}$, then $K\{M_p\}$ is the test space $\mathscr{K}_1$ of [9]. Again conditions (M), (N) and (P) are satisfied.

Example 4. Let $\{r_j\}$ be a sequence such that $0 < r_1 < r_2 < \ldots < r$ and $r_j \to r$. Set $M_p(t) = \exp(r_p|t|)$ for $t \in \boldsymbol{R}$. In this case $K\{M_p\} = \boldsymbol{H}_r$ as in [8], and $\{M_p\}$ satisfies conditions (M), (N) and (P).

In section II.4.2 of [3] it is shown that the sequence of norms

$$\|\varphi\|_{p,1} = \sup_{|\alpha| \leqslant p} \int M_p(x)\,|D^\alpha \varphi(x)|\,dx \qquad (p \geqslant 1) \tag{4}$$

generates the same locally convex topology on $K\{M_p\}$ as the sequence $\{\|\ \|_p\}$ given in (3). (Here $\int f(x)dx$ denotes the integral of f over S.) To obtain our first result we consider another sequence of norms. Note that since $m_{pp'}$ in condition (N) is summable over $\boldsymbol{R}^m$ and $m_{pp'}(x) \to 0$ as $|x| \to \infty$, we infer that $m_{pp'} \in L^2(\boldsymbol{R}^m)$. If $\varphi \in K\{M_p\}$ and $|\alpha| \leqslant p$, then

$$M_p(x)\,|D^\alpha \varphi(x)| \leqslant m_{pp'}(x)\,\|\varphi\|_{p'},$$

where p' is given by (N), so that $M_p D^\alpha \varphi$ is in $L^2(\boldsymbol{R}^m)$. Thus we may consider the sequence of norms given by

$$\|\varphi\|_{p,2} = \sup_{|\alpha| \leqslant p} \Big(\int \big(M_p(x)\,|D^\alpha \varphi(x)|\big)^2 dx\Big)^{1/2} \qquad (p \geqslant 1). \tag{5}$$

(Similar L^2-type norms are considered in Theorem 7 of I. 3.6 of [4].) First, we show that the sequence of norms in (5) is equivalent to the sequence of norms in (3).

LEMMA 5. *The sequence of norms $\{\|\ \|_p\}$ is equivalent to the sequence of norms $\{\|\ \|_{p,2}\}$, i.e., the two sequences generate the same locally convex topology on $K\{M_p\}$.*

Proof. Given p and a with $|a| \leqslant p$, we have, for $\varphi \in K\{M_p\}$,

$$\int M_p^2(x)\,|D^a\varphi(x)|^2\,dx \leqslant \sup_x M_{p'}^2(x)\,|D^a\varphi(x)|^2 \int m_{pp'}^2(x)\,dx,$$

where p' is given as in condition (N). Thus there is a constant $C_p > 0$ such that $\|\varphi\|_{p,2} \leqslant C_p \|\varphi\|_{p'}$. On the other hand, there is a constant A_p and a positive integer $q \geqslant p$ such that $\|\varphi\|_p \leqslant A_p \|\varphi\|_{q,1}$ (see [3], II. 4.2). Let q' correspond to q as in condition (N). Then we have, by the Cauchy-Schwartz inequality,

$$\int M_q(x)\,|D^a\varphi(x)|\,dx \leqslant \Big(\int m_{qq'}^2(x)\,dx\Big)^{1/2}\Big(\int M_{q'}^2(x)\,|D^a\varphi(x)|^2\,dx\Big)^{1/2} \quad \text{for } |a| \leqslant q.$$

Thus there is a constant B_p such that $\|\varphi\|_p \leqslant A_p\|\varphi\|_{q,1} \leqslant B_p\|\varphi\|_{q',2}$ and the lemma follows.

Remark. The equivalence of $\{\|\ \|_p\}$ and $\{\|\ \|_{p,2}\}$ is proved in Theorem 7 of I.3.6 of [4] with some additional assumptions on the $\{M_p\}$. (See equation (10) of I.3.6 in [4]; in particular, it is assumed that the M_p are infinitely differentiable.) From the lemma, these additional assumptions are not necessary.

We now give the first characterization of sequential convergence in $K\{M_p\}'$.

THEOREM 6. *Let $\{M_p\}$ satisfy conditions* (M), (N) *and* (P). *The following conditions are equivalent*:

(i) $T_n \to 0$ *weakly* (*strongly*) *in* $K\{M_p\}'$ (see [3], I.6.4);

(ii) *there exist a positive integer p and, for each multi-index a with $|a| \leqslant p$, a sequence $\{f_{a,n}\}_{n=1}^\infty \subseteq L^2(S)$ such that*

$$T_n = \sum_{|a|\leqslant p} (-1)^{|a|} D^a(M_p f_{a,n}) \quad \textit{and} \quad f_{a,n} \to 0 \quad \textit{in } L^2(S).$$

Proof. Suppose $T_n \to 0$ weakly in $K\{M_p\}'$. By I.6.4 of [3], there is a positive integer p such that

$$\sup\big\{|\langle T_n, \varphi\rangle| :\ \varphi \in K\{M_p\},\ \|\varphi\|_{p,2} \leqslant 1\big\} \to 0 \quad \text{as } n \to \infty. \tag{6}$$

In particular, there is a constant $B > 0$ such that $|\langle T_n, \varphi\rangle| \leqslant B\|\varphi\|_{p,2}$ for $\varphi \in K\{M_p\}$. Let Γ be the direct sum of a finite number (equal to the number of multi-indices a such that $|a| \leqslant p$) of copies of $L^2(S)$ and equip Γ with the norm

$$\|\{f_a\}_{|a|\leqslant p}\| = \sup_{|a|\leqslant p} \|f_a\|_2, \qquad \text{where } \|f_a\|_2 = \Big(\int |f_a(x)|^2\,dx\Big)^{1/2}.$$

Define a map θ from $K\{M_p\}$ into Γ by $\theta\colon \varphi \to \{M_p D^a\varphi\}_{|a|\leqslant p}$, and note that θ is one-one. Let Δ be the image of $K\{M_p\}$ under θ, $\overline{\Delta}$ the closure of Δ in Γ, and $\bar{\Delta}^{\perp}$ the orthogonal complement of $\bar{\Delta}$ in Γ. For each n, define a linear functional L_n on Δ by $\langle L_n, \theta(\varphi)\rangle = \langle T_n, \varphi\rangle$. Since

$$|\langle L_n, \theta(\varphi)\rangle| \leqslant B\|\varphi\|_{p,2} = B\|\theta(\varphi)\|,$$

L_n is continuous. We extend L_n to $\bar{\Delta}$ by continuity, and then to Γ by setting $\langle L_n, g\rangle = 0$ for $g \in \bar{\Delta}^{\perp}$. This extension, which we continue to denote by L_n, has the same norm as L_n over Δ. But, by (6), $\|L_n\| \to 0$ in Γ' as $n \to \infty$. By the Riesz Representation Theorem, for each n, there exist functions $\{f_{\alpha,n}\colon |\alpha| \leqslant p\} \subseteq L^2(S)$ such that, for each $G = \{g_\alpha\}_{|\alpha|\leqslant p} \in \Gamma$,

$$\langle L_n, G\rangle = \sum_{|\alpha|\leqslant p} \int f_{\alpha,n}(x) g_\alpha(x)\,dx \qquad \text{with } \|L_n\| = \sum_{|\alpha|\leqslant p} \|f_{\alpha,n}\|_2.$$

In particular, for $\varphi \in K\{M_p\}$,

$$\langle L_n, \theta(\varphi)\rangle = \langle T_n, \varphi\rangle = \sum_{|\alpha|\leqslant p} \int f_{\alpha,n}(x) M_p(x) D^\alpha \varphi(x)\,dx$$

or

$$T_n = \sum_{|\alpha|\leqslant p} (-1)^{|\alpha|} D^\alpha (M_p f_{\alpha,n}).$$

Since

$$\|L_n\| = \sum_{|\alpha|\leqslant p} \|f_{\alpha,n}\|_2 \to 0,$$

(ii) is established.

To show that (ii) implies (i) note that, for $\varphi \in K\{M_p\}$,

$$|\langle T_n, \varphi\rangle| \leqslant \sum_{|\alpha|\leqslant p} \int |f_{\alpha,n}(x)| M_p(x) |D^\alpha \varphi(x)|\,dx \leqslant \|\varphi\|_{p,2} \sum_{|\alpha|\leqslant p} \|f_{\alpha,n}\|_2$$

so that, by (ii), $T_n \to 0$ weakly in $K\{M_p\}'$.

Remark. The proof of Theorem 6 presented here differs from the proof of the characterization of convergent sequences in $\mathscr{D}'_K$ given in [5] (Theorem XXIII of Chapter III) or in [2] (Theorem 19, Section 5 of Chapter 3). The proofs in [5] and [2] seem to rely on using L^2-space methods to extend the linear functional L_n, and then obtain the fact that $L_n \to 0$ weakly in Γ'. By employing the result in I.6.4 of [3], we obtain immediately that actually $L_n \to 0$ strongly in Γ'. After making this observation, we see that it is really not important to use the L^2-space. That is, we could use the norms $\{\|\ \|_{p,1}\}$ (or $\{\|\ \|_p\}$) and let Γ be a direct sum of $L^1(S)$ (or $C(S)$) and obtain a representation as in (ii) with $f_{\alpha,n} \in L^\infty(S)$ and $\lim_n \|f_{\alpha,n}\|_\infty = 0$ (or $f_{\alpha,n}$ bounded measures with $\operatorname{var}(f_{\alpha,n}) \to 0$).

We now give a characterization of sequential convergence in $K\{M_p\}'$ in terms of regularizations. For this result we impose an additional condition on the sequence $\{M_p\}$. The sequence $\{M_p\}$ satisfies the following condition:

(F) Each M_p is finite valued, $M_p(x) = M_p(-x)$ for $x \in \boldsymbol{R}^m$, and, for each p, there are $p' > p$ and C_p such that

$$M_p(x+h) \leqslant C_p M_{p'}(x) M_{p'}(h) \quad \text{for } x, h \in \boldsymbol{R}^m.$$

Problems concerning convolution in $K\{M_p\}$-spaces have been treated in [7]. In particular, if condition (F) is satisfied, translation is continuous on $K\{M_p\}$ and regularizations can be formed (see Lemma 1 of [7] and III.3.1 of [3]). Recall that if $T \in \mathscr{D}'$ and $\varphi \in \mathscr{D}$, the regularization of T by φ is the function $T*\varphi$: $x \to \langle T, (\tau_{-x}\varphi)^{\vee}\rangle$, where $\tau_{-x}\varphi$: $y \to \varphi(y-x)$ and $\check{\varphi}$: $x \to \varphi(-x)$. Thus, if $T \in K\{M_p\}'$ and $\varphi \in \mathscr{D}$, then $T*\varphi$ is an infinitely differentiable function (see [5], Chapter VI, Theorem XI).

Before stating our result, we introduce some auxiliary spaces. For each positive integer p, let B_p be the vector space of all continuous complex-valued functions f on $\boldsymbol{R}^m$ such that

$$|f|_p = \sup\{|f(x)|/M_p(x):\ x \in \boldsymbol{R}^m\} < \infty.$$

We equip B_p with the norm $|\ |_p$, and note that B_p is a B-space under this norm.

THEOREM 7. *Let $\{M_p\}$ satisfy conditions* (M), (N) *and* (F). *The following conditions are equivalent*:

(i) $T_n \to 0$ *in* $K\{M_p\}'$ (*weakly or strongly*);

(ii) *condition* (ii) *of Theorem* 6;

(iii) *there is a positive integer q such that $T_n*\varphi \to 0$ in B_q for each $\varphi \in \mathscr{D}$*;

(iv) *there exist positive integers q and l and, for each multi-index α with $|\alpha| \leqslant l$, there exists a sequence $\{f_{\alpha,n}\} \subseteq B_q$ such that*

$$\lim_n f_{\alpha,n} = 0 \text{ in } B_q \quad \text{and} \quad T_n = \sum_{|\alpha|\leqslant l} D^\alpha f_{\alpha,n}.$$

Proof. Since each M_p is finite valued from condition (F), condition (N) implies condition (P) and Theorem 6 gives the equivalence of (i) and (ii).

Suppose (ii) holds. For $\varphi \in \mathscr{D}$, we have

$$\begin{aligned} |T_n*\varphi(x)| &= \Big|\sum_{|\alpha|\leqslant l}(-1)^{|\alpha|}\int M_p(y) f_{\alpha,n}(y) D^\alpha\varphi(x-y)\,dy\Big| \\ &\leqslant C_q M_q(x) \sum_{|\alpha|\leqslant l}\int M_q(t)\,|f_{\alpha,n}(x-t)|\,|D^\alpha\varphi(t)|\,dt \\ &\leqslant C_q M_q(x)\,\|\varphi\|_{q,2} \sum_{|\alpha|\leqslant l}\|f_{\alpha,n}\|_2, \end{aligned} \tag{7}$$

where $q = p'$ is given by condition (F), and the Cauchy-Schwartz inequality has been used. By (ii) and (7), $T_n*\varphi \to 0$ in B_q, and (iii) is established.

Suppose (iii) holds. To establish (iv) we apply Theorem 3 of [1] with the space B in this theorem equal to B_q as above. By the conclusion of this theorem, there are a positive integer l and sequences $\{f_n\}$ and $\{g_n\}$ from B_q such that $\lim f_n = \lim g_n = 0$ in B_q and

$$(8) \qquad \langle T_n, \varphi\rangle = \int f_n(x)\big(1-\Delta(x)\big)^l\varphi(x)\,dx + \int g_n(x)\varphi(x)\,dx \qquad \text{for } \varphi \in \mathscr{D}.$$

Since f_n and g_n belong to B_q, the map

$$\varphi \to \int f_n(x)\,\big(1-\Delta(x)\big)^l\varphi(x)\,dx + \int g_n(x)\varphi(x)\,dx$$

defines a continuous linear functional on $K\{M_p\}$ (see [3], II.4.2), and equation (8) shows this continuous linear functional agrees with T_n on the dense set $\mathscr{D}$ (see [3], II.2.5). Therefore, equation (8) is valid for $\varphi \in K\{M_p\}$ and (iv) follows.

Suppose (iv) holds. We may assume that $q \geqslant l$ in (iv) since the injections $B_j \to B_{j+1}$ are continuous. For $\varphi \in K\{M_p\}$,

$$\begin{aligned}|\langle T_n, \varphi\rangle| &\leqslant \sum_{|\alpha|<l} \int |f_{\alpha,n}(x) D^\alpha \varphi(x)|\,dx \\ &\leqslant \sup\{|f_{\alpha,n}(x)/M_q(x)|:\ x \in \boldsymbol{R}^m,\ |\alpha| \leqslant l\}\,\|\varphi\|_{q,1},\end{aligned}$$

so that $T_n \to 0$ weakly. That is, (i) holds and the result is established.

Remarks. Note that the spaces in Examples 1-4 satisfy condition (F) so that Theorem 7 is applicable to these spaces.

For $K\{M_p\} = \mathscr{S}$, the equivalence of (i) and (iv) is recorded in Theorem 56 in Chapter 3 of [2]. (See also the remark following Theorem VI of Chapter VII of [5].) No analogues of the regularization condition (iii) seems to be recorded, even for $\mathscr{S}$.

For $K\{M_p\} = H_r$ as in Example 4, the equivalence of (i) and (iv) is given in Theorem 3 of [8].

We conclude by mentioning that it might be possible to alter condition (iii) of Theorem 7 somewhat. Note that $B_q \subseteq B_{q+1}$ with the injection continuous. Thus, if we set

$$B = \bigcup_{q \geqslant 1} B_q,$$

B may be supplied with the inductive limit topology from the $\{B_q\}$. If this inductive limit is regular (i.e., a set $A \subseteq B$ is bounded iff A is contained in some B_q and bounded in B_q), B will be sequentially complete, and, by the proof of (iii) implies (iv) above, we can replace condition (iii) with the condition

(iii)′ *$T_n * \varphi \to 0$ in B for each $\varphi \in \mathscr{D}$.*

(See the statement of Theorem 3 in [1].) One possible way of showing that the inductive limit is regular would be to show that, for each q, there is a $q' > q$ such that the injection $B_q \to B_{q'}$ is compact (see [6]); however, we have not been able to establish this.

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