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ON CONFORMAL COLLINEATIONS

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1. Introduction. An n-dimensional ($n \geqslant 4$) Riemannian manifold (whose metric g_{ij} need not be definite) is called *conformally symmetric* [2] if its Weyl conformal curvature tensor

$$(1) \qquad C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}) + \frac{R}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{ik}g_{hj})$$

is parallel, i.e., if $C_{hijk,l} = 0$.

Here and in the sequel we denote by R_{hijk}, R_{ij} and R the curvature tensor, Ricci tensor and scalar curvature, respectively, while the comma stands for covariant differentiation with respect to g.

Clearly, the class of conformally symmetric manifolds contains all locally symmetric as well as all conformally flat manifolds of dimension $n \geqslant 4$.

The existence of essentially conformally symmetric manifolds, i.e., conformally symmetric manifolds which are neither conformally flat nor locally symmetric, has been established in [15] (see also [16]) as follows:

EXAMPLE 1. Let M denote the Euclidean n-space ($n \geqslant 4$) endowed with the metric $g_{\lambda\mu}$ (λ, $\mu = 1, 2, \ldots, n$) given by

$$(2) \qquad \begin{gathered} g_{\lambda\mu}dx^\lambda dx^\mu = Q(dx^1)^2 + k_{ij}dx^i dx^j + 2dx^1 dx^n, \\ Q = (Bk_{ij} + c_{ij})x^i x^j, \end{gathered}$$

where i, $j = 2, 3, \ldots, n-1$, $[k_{ij}]$ is a symmetric and non-singular matrix consisting of constants, $[c_{ij}]$ is a symmetric and non-zero matrix of constants satisfying $k^{ij}c_{ij} = 0$ with $[k^{ij}] = [k_{ij}]^{-1}$, and B is a non-constant function of x^1 only. Then M is essentially conformally symmetric.

It is easy to check (cf. Lemma 1, equation (10)) that for every conformally symmetric manifold the condition

$$(3) \qquad R_{ij,k} - R_{ik,j} = \frac{1}{2(n-1)}(R_{,k}g_{ij} - R_{,j}g_{ik})$$

holds.

An n-dimensional ($n \geqslant 3$) Riemannian manifold (with a possibly indefinite metric) is said to be *nearly conformally symmetric* (NCS for short) if its Ricci tensor satisfies condition (3).

Riemannian manifolds ($n \geqslant 4$) satisfying (3) are also said to have *harmonic Weyl* (*conformal curvature*) *tensor* ([1], p. 440).

Any conformally symmetric manifold is therefore nearly conformally symmetric. Moreover, equation (3) shows that every n-dimensional ($n \geqslant 3$) manifold of harmonic curvature ($R_{ij,k} = R_{ik,j}$) is also nearly conformally symmetric.

An NCS-manifold ($n \geqslant 4$) is called *almost conformally symmetric* if it is neither conformally symmetric nor of harmonic curvature.

The existence of non-trivial NCS-manifolds can be stated [17] as follows:

EXAMPLE 2. Let $M = R^{n-1} \times R^1_+$ ($n \geqslant 5$) be endowed with the metric $g_{\lambda\mu}(\lambda, \mu = 1, 2, \ldots, n)$ given by

(4) $$g_{\lambda\mu} dx^\lambda dx^\mu = \left((n-1)x^n\right)^{2/(n-1)} f_{ij} dx^i dx^j + (dx^n)^2,$$

where $i, j = 1, 2, \ldots, n-1$, and $f_{ij}(x^1, \ldots, x^{n-1})$ is an arbitrary non-flat Ricci-flat metric (which evidently exists if $n \geqslant 5$). Then:

(i) (M, g) is almost conformally symmetric.

(ii) For any (smooth) function p depending on x^n only, the metric $\bar{g} = (\exp 2p)g$ is nearly conformally symmetric. Moreover, as one can easily verify (cf. [17]), the metric

$$\bar{g} = (\exp 2p)g \quad \text{with } p = \frac{1}{n-1} \ln x^n$$

is of harmonic curvature and is neither conformally symmetric nor Ricci--symmetric ($R_{ij,k} = 0$).

Let M be a Riemannian manifold with a (not necessarily definite) metric g. By an *infinitesimal conformal motion* on M we shall mean a vector field v on M such that $L_v g_{ij} = 2A g_{ij}$, where L_v denotes the Lie derivative with respect to v (i.e., $L_v g_{ij} = v_{i,j} + v_{j,i}$) and A is a function on M (clearly, $A = n^{-1} v^r{}_{,r}$)(¹).

If $A = \text{const}$, then the conformal motion is said to be *homothetic*. If $L_v g_{ij} = 0$ everywhere on M, then v is called a *motion* or an *isometry*.

It is well known that for a conformal motion the condition

(5) $$L_v \begin{Bmatrix} h \\ i\ j \end{Bmatrix} = \delta^h_i A_j + \delta^h_j A_i - A^h g_{ij}$$

holds, where $A^h = g^{hr} A_r$.

A vector field v on M is said to be a *conformal collineation* [19] if it satisfies condition (5).

(¹) Since all motions and collineations appearing in this paper are infinitesimal, we shall omit the word "infinitesimal".

It is easy to check [19] that $A_j = n^{-1}(v^r{}_{,r})_{,j}$. Hence A_j is a gradient vector field.

Condition (5) is equivalent ([8], p. 30) to

$$a_{ij,k} = 2A_k g_{ij}, \tag{6}$$

a_{ij} being the Lie derivative of g_{ij} with respect to v.

Clearly, every conformal motion is a conformal collineation. The converse statement fails in general ([19], see also Example 3).

If

$$L_v\begin{Bmatrix} h \\ i\ j \end{Bmatrix} = 0 \text{ everywhere on } M,$$

then v is said to be an *affine collineation.*

Every homothetic conformal motion is necessarily an affine collineation, but the converse statement is not in general true.

Conformal collineations have been studied by many authors. In particular, Gębarowski [7] proved by a straightforward computation (see also [17], Corollary 7) the following result:

THEOREM 1. *Let M be an n-dimensional ($n \geqslant 3$) NCS-manifold (with a possibly indefinite metric). If M is not of harmonic curvature and admits a conformal collineation, then this collineation is a conformal motion.*

As an immediate consequence of Theorem 1, we have

COROLLARY 1. *Let M be an n-dimensional ($n \geqslant 3$) conformally flat manifold (with a possibly indefinite metric). If its scalar curvature is not a constant and M admits a conformal collineation, then this collineation is a conformal motion.*

The present paper deals (in fact) with conformal collineations on NCS-manifolds. Since Theorem 1 and Example 4 give sufficient information on conformal collineations on NCS-manifolds with non-constant scalar curvature, we shall restrict our consideration to manifolds of harmonic curvature.

More precisely, Section 3 of this paper deals with conformal collineations on Einstein manifolds, while Section 4 contains some results concerning conformal collineations on conformally symmetric manifolds.

Obviously, all manifolds considered in the above-mentioned sections are of harmonic curvature. Moreover, except for certain classes of conformally flat manifolds, they are all semi-Ricci-symmetric, i.e., their Ricci tensor satisfies the condition (cf. Lemma 9)

$$R_{ri}R^r{}_{jkl} + R_{rj}R^r{}_{ikl} = 0. \tag{7}$$

Finally, the last section of this paper is concerned with conformal collineations on analytic semi-Ricci-symmetric manifolds of harmonic curvature (without any additional assumptions).

All manifolds under consideration are assumed to be connected and of class C^∞ or analytic. Their Riemannian metrics, unless stated otherwise, are not assumed to be definite.

2. Preliminaries. In the sequel we shall need the following results:

LEMMA 1. *The Weyl conformal curvature tensor satisfies the well-known equations*:

$$C_{hijk} = -C_{ihjk} = -C_{hikj} = C_{jkhi}, \tag{8}$$

$$C_{hijk}+C_{hjki}+C_{hkij} = 0, \quad C^r{}_{ijr} = C^r{}_{irk} = C^r{}_{rjk} = 0, \tag{9}$$

$$C^r{}_{ijk,r} = \frac{n-3}{n-2}\left(R_{ij,k}-R_{ik,j}-\frac{1}{2(n-1)}(R_{,k}g_{ij}-R_{,j}g_{ik})\right). \tag{10}$$

LEMMA 2 ([13], Lemma 2). *If c_j, p_j and B_{hijk} are numbers satisfying*

$$c_l B_{hijk}+p_h B_{lijk}+p_i B_{hljk}+p_j B_{hilk}+p_k B_{hijl} = 0,$$

$$B_{hijk} = B_{jkhi} = -B_{hikj}, \quad B_{hijk}+B_{hjki}+B_{hkij} = 0,$$

then each $b_j = c_j+2p_j$ is zero or each B_{hijk} is zero.

LEMMA 3. *If a Riemannian manifold admits a conformal collineation, then*

$$LR^h{}_{ijk} = \delta^h_j A_{i,k}-\delta^h_k A_{i,j}+A^h{}_{,j}g_{ik}-A^h{}_{,k}g_{ij}, \tag{11}$$

$$LR_{ij} = (2-n)A_{i,j}-Gg_{ij}, \quad G = A^r{}_{,r} = g^{rs}A_{r,s}, \tag{12}$$

$$LR^h{}_j = (2-n)A^h{}_{,j}-a^h{}_r R^r{}_j-G\delta^h_j, \quad LR = 2(1-n)G-a^{rs}R_{rs}, \tag{13}$$

where $a^h{}_j = g^{hr}a_{rj}$ and $a_{ij} = Lg_{ij}$.

The proof is trivial.

LEMMA 4. *Let M be a Riemannian manifold admitting a conformal collineation. If the scalar curvature of M is a constant, then*

$$LC^h{}_{ijk} = \frac{1}{n-2}(a_{ik}R^h{}_j-a_{ij}R^h{}_k+g_{ij}a^h{}_r R^r{}_k-g_{ik}a^h{}_r R^r{}_j) + \frac{2}{n-2}G(\delta^h_k g_{ij}-\delta^h_j g_{ik})+\frac{R}{(n-1)(n-2)}(\delta^h_k a_{ij}-\delta^h_j a_{ik}), \tag{14}$$

$$(LC^h{}_{ijk})_{,l} = \frac{1}{n-2}(a_{ik}R^h{}_{j,l}-a_{ij}R^h{}_{k,l}+g_{ij}a^h{}_r R^r{}_{k,l}-g_{ik}a^h{}_r R^r{}_{j,l}) + \frac{2}{n-2}\left(G_{,l}+\frac{1}{n-1}RA_l\right)(\delta^h_k g_{ij}-\delta^h_j g_{ik}). \tag{15}$$

Proof. Equation (14) is a consequence of (1), (11) and (13). Equation (15) follows from (14) and (6).

LEMMA 5. *If a Riemannian manifold M admits a conformal collineation, then T_{ij} given by*

$$T_{ij} = a_{ij}-\frac{1}{n}ag_{ij},$$

where $a = g^{rs}a_{rs}$, is symmetric and parallel on M.

Proof. The assertion is an immediate consequence of (6).

The following lemma seems to be well known:

LEMMA 6. *Let p_i denote a null parallel vector field on M. Then the equations*

$$p_r R^r{}_{ijk} = 0, \quad p_r R^r{}_j = 0, \quad p^r p_r = 0$$

hold.

LEMMA 7 ([3], Theorem 2). *Let M be a conformally symmetric manifold with a positive definite metric. Then M is conformally flat or locally symmetric.*

LEMMA 8. *Let M be a conformally symmetric manifold admitting a symmetric parallel tensor field $a_{ij} \neq cg_{ij}$ (c = const).*

(i) *If the metric of M is definite, then M is locally symmetric.*

(ii) *If M is non-locally symmetric (and, in consequence, its metric must be indefinite), then for each point $x \in M$ such that $R_{ij,k}(x) \neq 0$ there exists a non-trivial null parallel vector field p_j on some neighbourhood U of x satisfying*

$$R_{ij,k} = f p_i p_j p_k, \quad a_{ij} = \frac{1}{n} a g_{ij} + e p_i p_j, \tag{16}$$

where $e = \pm 1$, $a = g^{rs} a_{rs}$ and $f \neq 0$ is a function on U.

Proof. Using Lemma 3 of [12] we can follow step by step the proof of Lemma 4 in [12] to show the assertion.

LEMMA 9. *Let M be a conformally symmetric manifold. If M is not conformally flat, then its Ricci tensor satisfies* (7).

Proof. If M is non-locally symmetric, then (7) is a consequence of Theorem 9 in [4]. Otherwise (7) is trivial.

LEMMA 10. *Let M be a conformally symmetric manifold. If M is not conformally flat, then its scalar curvature is a constant.*

Proof. This result follows from Theorem 7 of [4].

LEMMA 11. *Let M be a conformally symmetric manifold admitting a symmetric parallel tensor field $a_{ij} \neq cg_{ij}$ (c = const). Then M is Ricci-recurrent, i.e., its Ricci tensor satisfies the condition*

$$R_{hl} R_{ij,k} = R_{hl,k} R_{ij} \tag{17}$$

everywhere on M.

Proof. Clearly, it is sufficient to prove (17) at points (if they exist) where $R_{ij,k} \neq 0$. Let $x \in M$ be such that $R_{ij,k}(x) \neq 0$. Then, by Lemma 8, there exists a non-trivial null parallel vector field p_j on some neighbourhood U of x satisfying conditions (16).

If M is not conformally flat, then, by Lemma 9, we have

$$R_{ri,l} R^r{}_{jhk} + R_{rj,l} R^r{}_{ihk} + R_{ri} R^r{}_{jhk,l} + R_{rj} R^r{}_{ihk,l} = 0,$$

which, because of (1) and Lemma 10, can be written in the form

$$R_{ik}R_{jh,l}-R_{hi}R_{jk,l}+R_{jk}R_{ih,l}-R_{jh}R_{ik,l}+g_{jh}R_{ri}R^{r}{}_{k,l}$$
$$-g_{jk}R_{ri}R^{r}{}_{h,l}+g_{ih}R_{rj}R^{r}{}_{k,l}-g_{ik}R_{rj}R^{r}{}_{h,l}$$
$$+(n-2)(R_{ri,l}R^{r}{}_{jhk}+R_{rj,l}R^{r}{}_{ihk})=0.$$

Substituting (16) into the last equation and making use of Lemma 6, we obtain easily on U

(18) $$p_jp_hR_{ik}-p_jp_kR_{ih}+p_ip_hR_{jk}-p_ip_kR_{jh}=0.$$

Since p_j is non-zero on U, we may choose a vector u^i at x such that $u^rp_r=1$. Transvecting (18) with u^ju^h and putting $q_j=u^rR_{rj}$ and $q=u^ru^sR_{rs}$, we get

$$R_{ik}=q_ip_k-q_kp_i+qp_ip_k,$$

which, by further transvection with u^i, implies $q_j=qp_j$. Hence $R_{ij}=qp_ip_j$. The last result, together with (16), shows that (17) holds at x.

Suppose now that M is conformally flat. Then, in view of (1) and Lemma 6, we have on U

$$p_kR_{ij}-p_jR_{ik}=\frac{1}{n-1}R(p_kg_{ij}-p_jg_{ik}),$$

whence, by transvection with p^k, we get $Rp_ip_j=0$. Hence $R=0$ and, consequently, $p_kR_{ij}=p_jR_{ik}$. But the last result yields $R_{ij}=hp_ip_j$, which, together with (16), completes the proof.

Remark 1. With the help of Lemma 11 we shall prove in a subsequent paper the following result:

Let M be an analytic non-locally symmetric conformally symmetric manifold admitting a symmetric parallel tensor field $a_{\lambda\mu}\neq cg_{\lambda\mu}$ ($c=$ const, $\lambda,\ \mu=1,2,\dots,n$).

(i) Then for each point $x\in M$ there exists a coordinate system in a neighbourhood of x such that the metric of M takes the form (2), where $[k_{ij}]$ ($i,j=2,3,\dots,n-1$) is a symmetric and non-singular matrix consisting of constants, $[c_{ij}]$ is a symmetric matrix of constants satisfying $k^{ij}c_{ij}=0$ with $[k^{ij}]=[k_{ij}]^{-1}$, and B is a non-constant function of x^1 only.

(ii) In this coordinate system the tensor field $a_{\lambda\mu}$ is of the form

(19) $$a_{\lambda\mu}=Cg_{\lambda\mu}+e\delta_\lambda^1\delta_\mu^1,$$

where $C=$ const and $e=\pm1$.

(iii) Conversely, given $[k_{ij}]$ and $[c_{ij}]$ with properties stated in (i) and a non-constant function B of x^1 only. Then formulae (2) define a non-locally symmetric conformally symmetric metric. Moreover, every tensor field $a_{\lambda\mu}$ of the form

$$a_{\lambda\mu}=C_1g_{\lambda\mu}+C_2\delta_\lambda^1\delta_\mu^1\quad (C_1,\ C_2=\text{const},\ C_2\neq0)$$

is parallel and is not a multiple of $g_{\lambda\mu}$.

Remark 2. A tensor field $T_{i_1\dots i_p}$ on M is said to be *recurrent* if

$$T_{j_1\dots j_p} T_{i_1\dots i_p,k} = T_{j_1\dots j_p,k} T_{i_1\dots i_p}. \tag{20}$$

In particular, every parallel tensor field on M is recurrent.

Condition (20) states that at any point $x \in M$ such that $T(x) \neq 0$ there exists a (unique) covariant vector b (called the *recurrence vector* of T) which satisfies

$$T_{i_1\dots i_p,k} = b_k T_{i_1\dots i_p}. \tag{21}$$

The above definition of recurrency differs slightly from the classical one (given by (21)). Obviously, both definitions are equivalent on the subset of M where T does not vanish.

Throughout this paper, by a *recurrent* (*Ricci-recurrent* (cf. [11])) *manifold* we shall mean a Riemannian manifold whose curvature tensor (Ricci tensor) is recurrent.

Since every conformally flat Ricci-recurrent manifold is recurrent, Lemma 11 implies

COROLLARY 2 ([9], [10]). *If a conformally flat manifold M ($n \geqslant 4$) admits a symmetric parallel tensor field $a_{ij} \neq cg_{ij}$ (c = const), then M is recurrent.*

LEMMA 12. *Let M be a conformally symmetric manifold. If v denotes a conformal collineation on M, then*

$$(L_v C^h{}_{ijk})_{,l} = 0. \tag{22}$$

Proof. Clearly, we may assume that neither M is conformally flat nor v is a conformal motion.

Let $x \in M$ (if it exists) be such that $R_{ij,k}(x) \neq 0$. Then, by Lemmas 5 and 8, there exists a non-trivial null parallel vector field p_j on some neighbourhood of x which satisfies (16).

Using Lemma 10 and substituting (16) into (15), we obtain easily

$$(L_v C^h{}_{ijk})_{,l} = \frac{2}{n-2}\left(G_{,l} + \frac{1}{n-1} R A_l\right)(\delta^h_k g_{ij} - \delta^h_j g_{ik}), \tag{23}$$

which, because of $(L_v C^r{}_{ijr})_{,l} = 0$, leads immediately to (22).

If now $R_{ij,k}(x) = 0$, then, by (15), equation (23) holds at x. But (23), by an argument similar to the above one, implies (22). Hence (22) is satisfied at any point of M. This completes the proof.

3. Conformal collineations on Einstein manifolds. In this section we shall discuss conformal collineations on Einstein manifolds.

PROPOSITION 1. *Let M be an n-dimensional ($n \geqslant 3$) Einstein manifold. If M is not Ricci-flat and admits a conformal collineation, then this collineation is a conformal motion.*

Proof. Since $R_{ij} = n^{-1}Rg_{ij}$, (12) yields

$$A_{i,j} = \frac{1}{2-n}Gg_{ij} + \frac{1}{n(2-n)}Ra_{ij}, \tag{24}$$

whence, by contraction with g^{ij}, we have

$$2(1-n)G = \frac{1}{n}Ra, \tag{25}$$

where $a = g^{rs}a_{rs}$.

But (25), in view of (6), implies

$$G_{,j} = \frac{1}{1-n}RA_j,$$

which, together with (24), gives

$$A_{i,jk} = \frac{1}{n(1-n)}RA_k g_{ij}.$$

Hence

$$A_r R^r{}_{ijk} = \frac{1}{n(n-1)}R(A_k g_{ij} - A_j g_{ik}). \tag{26}$$

Differentiating (26) covariantly and taking (24) into account, we get

$$\begin{aligned} A_r R^r{}_{ijk,l} &+ \frac{1}{2-n}GR_{lijk} + \frac{1}{n(2-n)}Ra_{rl}R^r{}_{ijk} \\ &= \frac{1}{n(n-1)(2-n)}R\left(G(g_{ij}g_{kl} - g_{ik}g_{jl}) + \frac{1}{n}R(a_{kl}g_{ij} - a_{jl}g_{ik})\right). \end{aligned} \tag{27}$$

On the other hand, condition (6) yields

$$a_{rj}R^r{}_{ikl} + a_{ri}R^r{}_{jkl} = 0. \tag{28}$$

Interchanging in (27) the indices i and l, adding the resulting equations to (27) and making use of (28), we obtain easily

$$A_r R^r{}_{ijk,l} + A_r R^r{}_{ljk,i} = \frac{1}{n^2(n-1)(2-n)}R^2(a_{kl}g_{ij} - a_{jl}g_{ik} + a_{ki}g_{lj} - a_{ji}g_{lk}).$$

Contracting the last equation with g^{ij} and taking into consideration the obvious formulas

$$A_r R^r{}_{k,l} = 0 \quad \text{and} \quad A^r R^s{}_{krl,s} = A^r(R_{kr,l} - R_{kl,r}) = 0,$$

we get $R(a_{kl} - n^{-1}ag_{kl}) = 0$, which, since $R = \text{const} \neq 0$, completes the proof.

As an immediate consequence of (25) and (24), we have

COROLLARY 3. *If an n-dimensional* ($n \geqslant 3$) *Ricci-flat manifold admits a conformal collineation, then the vector field* A_j *is parallel and the conformal collineation is therefore a special curvature one* [8], *i.e.*,

$$\left(L_v \left\{ {h \atop i\ j} \right\}\right)_{,k} = 0.$$

Remark 3. Proposition 1 seems to belong to the folklore. Since the sequel of this paper requires equations (24)–(27) as well as Proposition 1 itself, we have included its proof for completeness.

PROPOSITION 2. *Let M be an n-dimensional* ($n \geqslant 3$) *locally symmetric Einstein manifold. If M is not Ricci-flat and admits a conformal collineation, then M is of constant curvature or the conformal collineation is an isometry.*

Proof. Substituting (25) and $a_{ij} = n^{-1} a g_{ij}$ into (27), we obtain easily $RaS_{lijk} = 0$, where

$$S_{hijk} = R_{hijk} - \frac{1}{n(n-1)} R(g_{ij} g_{hk} - g_{ik} g_{hj}). \tag{29}$$

Since M is locally symmetric by assumption, S is parallel on M. Therefore, if S vanishes at one point, then it vanishes everywhere on M. The last remark completes the proof.

As a consequence of (5) and (24), we have

COROLLARY 4. *Let M be an n-dimensional* ($n \geqslant 3$) *Einstein manifold. If M admits a special curvature collineation, then M is Ricci-flat or the curvature collineation reduces to an isometry.*

An n-dimensional ($n \geqslant 3$) Einstein manifold is said to be a *super-Einstein* one if its curvature tensor satisfies $R^{hlm}{}_i R_{hlmj} = w g_{ij}$ for some function w. It is well known that $w = \text{const}$ if $3 \leqslant \dim M \neq 4$.

THEOREM 2. *Let M be an n-dimensional* ($n \geqslant 3$) *super-Einstein manifold with a positive definite metric. If M is not Ricci-flat and admits a conformal collineation, then M is of constant curvature or the conformal collineation is an isometry.*

Proof. Equation (26) can be written in the form

$$A^r R_{kjir} = \frac{1}{n(n-1)} R(A_k g_{ij} - A_j g_{ik}).$$

Transvecting the last result with $R^{kji}{}_t$ and using the definition of a super-Einstein manifold, we obtain

$$\left(w - \frac{2}{n^2(n-1)} R^2\right) A_t = 0. \tag{30}$$

Since the Einstein metric is analytic in suitable coordinate systems (see [6]), (30) implies

$$w = \frac{2}{n^2(n-1)} R^2 \quad \text{or} \quad A_j = 0.$$

In the first case we have $S^{hijk} S_{hijk} = 0$, where S is defined by (29). Hence M is of constant curvature. If $A_j = 0$, then (24) implies $a_{ij} = 0$. This completes the proof.

4. Conformal collineations on conformally symmetric manifolds.

THEOREM 3. *Let M be a non-conformally flat conformally symmetric manifold. If M admits a conformal collineation, then this collineation is an affine one.*

Proof. Applying to the formula ([21], p. 161)

$$LC^h{}_{ijk,l} = (LC^h{}_{ijk})_{,l} + C^r{}_{ijk} L\left\{ {h \atop r\ l} \right\} - C^h{}_{rjk} L\left\{ {r \atop i\ l} \right\}$$
$$- C^h{}_{irk} L\left\{ {r \atop j\ l} \right\} - C^h{}_{ijr} L\left\{ {r \atop k\ l} \right\}$$

expressions (5) and (22), we get

$$(31) \qquad g_{hl} A_r C^r{}_{ijk} - 2A_l C_{hijk} - A_h C_{lijk} - A_i C_{hljk} - A_j C_{hilk}$$
$$- A_k C_{hijl} + g_{il} A^r C_{hrjk} + g_{jl} A^r C_{hirk} + g_{kl} A^r C_{hijr} = 0,$$

which, by contraction with g^{hl} and making use of Lemma 1, yields $A_r C^r{}_{ijk} = 0$. But the last result reduces (31) to the form

$$(32) \qquad 2A_l C_{hijk} + A_h C_{lijk} + A_i C_{hljk} + A_j C_{hilk} + A_k C_{hijl} = 0.$$

Suppose that there exists a point $x \in M$ such that $A_j(x) \neq 0$. Then, by (32) and Lemma 2 (with $c_l = 2A_l$), we obtain $C_{hijk}(x) = 0$, which, since C is parallel, extends to the whole of M — a contradiction. Hence $A_j = 0$ everywhere on M, which completes the proof.

COROLLARY 5. *Let M be a non-conformally flat conformally symmetric manifold. If M admits a conformal motion, then this motion is necessarily homothetic.*

Proof. Corollary 5 follows immediately from Theorem 3 and the definition of a conformal motion (cf. [16], Theorem).

COROLLARY 6. *Let M be a non-conformally flat conformally symmetric manifold. If M is not Ricci-recurrent and admits a conformal collineation, then this collineation is necessarily a homothetic conformal motion.*

Proof. Theorem 3 shows that $a_{ij,k} = 0$ (cf. equation (6)). If the conformal collineation were not a homothetic conformal motion, then M would admit a symmetric parallel tensor field $a_{ij} \neq c g_{ij}$ (c = const). The assertion is now a consequence of Lemma 11.

Remark 4. The metric described in Example 1 is Ricci-recurrent. The existence of non-Ricci-recurrent essentially conformally symmetric manifolds has been established in Theorem 2 of [5].

As an immediate consequence of Lemma 5 and Corollaries 2 and 6, we have

COROLLARY 7. *Let M be a non-Ricci-recurrent conformally symmetric manifold. If M admits a conformal collineation, then this collineation is necessarily a conformal motion.*

COROLLARY 8. *Let M be an n-dimensional* ($n \geqslant 4$) *non-flat conformally flat manifold with a positive definite metric. If M admits a conformal collineation which is not a conformal motion, then M is a non-Einsteinian locally symmetric manifold.*

Proof. The assertion follows from Lemma 8 and Proposition 1.

PROPOSITION 3. *Let M be a compact orientable non-conformally flat conformally symmetric manifold with a positive definite metric. If M admits a conformal collineation, then M is locally symmetric and the conformal collineation is necessarily an isometry.*

Proof. By Lemma 7, M is locally symmetric. Moreover, Theorem 3 states that the conformal collineation is an affine collineation. But, as has been shown by Yano ([21], Theorem 5.1, p. 222), every affine collineation reduces on a compact orientable manifold with a positive deflnite metric to an isometry. The last remark completes the proof.

5. Some examples. Let M be an analytic conformally flat manifold ($n \geqslant 4$) admitting an *essentially conformal collineation*, i.e., a conformal collineation which is neither a conformal motion nor an affine one. By Corollary 2, M is recurrent. On the other hand, using Lemma 6 and the second Bianchi identity, it is not hard to check that the metric of an analytic non-locally symmetric conformally flat manifold ($n \geqslant 4$), admitting a non-trivial parallel vector field w on some coordinate neighbourhood U, must be indefinite, its scalar curvature must vanish everywhere and w is necessarily null. Moreover, if $\bar{w}$ is another non-trivial parallel vector field on U, then $\bar{w} = cw$ (c = const). Hence, M cannot be a special recurrent manifold ([18], pp. 156 and 165).

Suppose now that M is non-locally symmetric and that its curvature tensor does not vanish at any point. Then, in view of (21), M is recurrent in the sense of Ruse and Walker (cf. [18] and [20]). Moreover, since M is conformally flat and non-special recurrent, its general metric form can be written as

$$ds^2 = Q(dx^1)^2 + k_{ij}dx^i dx^j + 2dx^1 dx^n, \qquad Q = Bk_{ij}x^i x^j,$$

$[k_{ij}]$ being a symmetric and non-singular matrix of constants ($i, j = 2, \ldots, n-1$), and $B \neq 0$ is a non-constant function of x^1 only ([18], p. 176). The metric just described is non-decomposable (cf. [10] and [18], p. 181).

Now we are going to show the existence of conformally flat manifolds admitting essentially conformal collineations.

EXAMPLE 3. Let M denote the Euclidean n-space ($n \geqslant 4$) endowed with the indefinite metric given by the first equation of (2), where

$$Q = Bk_{ij}x^i x^j, \quad B = \tfrac{1}{4} + n\exp(-2x^1)$$

and $[k_{ij}]$ is as above. Then:

(i) M is a non-special recurrent conformally flat space.

(ii) The vector field v given by

$$(33) \quad v^1 = \frac{1}{n}\exp(x^1), \quad v^i = \frac{1}{2n}\exp(x^1)x^i, \quad v^n = bx^1 - \frac{1}{4n}\exp(x^1)k_{ij}x^i x^j,$$

where $b = \text{const} \neq 0$, is an essentially conformal collineation.

(iii) If $b = 0$, then v is a non-homothetic conformal motion.

Proof. In view of the previous argumentation, (i) is obvious.

In the metric considered, the only non-vanishing Christoffel symbols are those related to

$$(34) \qquad \begin{Bmatrix} i \\ 1\;1 \end{Bmatrix} = -Bx^i, \quad \begin{Bmatrix} n \\ 1\;1 \end{Bmatrix} = \frac{1}{2}B_{.1}k_{ij}x^i x^j, \quad \begin{Bmatrix} n \\ 1\;i \end{Bmatrix} = Bk_{ij}x^j,$$

where the dot denotes partial differentiation with respect to coordinates.

Using (33) and (34), it is easy to check that the equations

$$(35) \qquad a_{\lambda\mu} = \frac{1}{n}\exp(x^1)g_{\lambda\mu} + ep_\lambda p_\mu, \quad a_{\lambda\mu,\varrho} = \frac{1}{n}\exp(x^1)\delta^1_\varrho g_{\lambda\mu}$$

hold, where

$$\lambda, \mu, \varrho = 1, 2, \ldots, n, \quad a_{\lambda\mu} = L_v g_{\lambda\mu} = v_{\lambda,\mu} + v_{\mu,\lambda},$$

$$p_\lambda = \sqrt{2eb}\,\delta^1_\lambda \quad \text{and} \quad e = \pm 1.$$

Hence, by (35),

$$a_{\lambda\mu,\varrho} = 2A_\varrho g_{\lambda\mu}, \quad \text{where } A_\lambda = \frac{1}{2n}\exp(x^1)\delta^1_\lambda.$$

The last result shows that v is a conformal collineation. If $b \neq 0$, then, because of (35), v is neither a conformal motion nor an affine collineation.

If $b = 0$, then p_λ vanishes. By (35), v is therefore a non-homothetic conformal motion. This completes the proof.

As an immediate consequence of Example 3, we have

COROLLARY 9. *For each $n \geqslant 4$ there exist n-dimensional non-special recurrent conformally flat manifolds which admit essentially conformal collineations.*

EXAMPLE 4. Let $M = R^{n-1} \times R^1_+$ ($n \geqslant 5$) be endowed with metric (4), where f_{ij} has the properties described in Example 2. It is easy to check that the vector field v given by

$$v^i = 0, \quad v^n = \left((n-1)x^n\right)^{1/(n-1)}$$

satisfies the condition

$$L_v g_{\lambda\mu} = v_{\lambda,\mu} + v_{\mu,\lambda} = 2\left((n-1)x^n\right)^{(2-n)/(n-1)} g_{\lambda\mu}.$$

Hence, by the last example, we have

COROLLARY 10. *For each $n \geqslant 5$ there exist n-dimensional almost conformally symmetric manifolds which admit non-homothetic conformal motions.*

6. Conformal collineations on semi-Ricci-symmetric manifolds of harmonic curvature.

LEMMA 13. *Let M, $\dim M \geqslant 3$, be of harmonic curvature satisfying condition (7). If M is analytic and admits a conformal collineation which is not affine, then the equation*

$$R_{ri}R^r{}_j = \frac{1}{n-1} RR_{ij} + \frac{1}{n} R^{rs}R_{rs}g_{ij} - \frac{1}{n(n-1)} R^2 g_{ij} \tag{36}$$

holds.

Proof. Applying to

$$R_{ij,k} = R_{ik,j} \tag{37}$$

the formula ([21], p. 16)

$$LR_{ij,k} = (LR_{ij})_{,k} - R_{rj}L\left\{{r \atop i\ k}\right\} - R_{ir}L\left\{{r \atop j\ k}\right\}$$

and using (5) and (12), we obtain

$$(n-2)A_rR^r{}_{ijk} - A_kR_{ij} + A_jR_{ik} + G_{,j}g_{ik} - G_{,k}g_{ij} + g_{ik}A_rR^r{}_j - g_{ij}A_rR^r{}_k = 0, \tag{38}$$

whence, by contraction with g^{ij}, we have

$$G_{,j} = \frac{1}{1-n} RA_j.$$

But the last result, together with (38), yields

$$(n-2)A_rR^r{}_{ijk} = A_kR_{ij} - A_jR_{ik} + \frac{1}{n-1} R(A_jg_{ik} - A_kg_{ij}) + g_{ij}A_rR^r{}_k - g_{ik}A_rR^r{}_j. \tag{39}$$

On the other hand, condition (7) implies

$$R^s{}_i A_r R^r{}_{kjs} + R^s{}_j A_r R^r{}_{kis} = 0,$$

which, because of (39), can be written as

$$(40)\quad A_j R_{rk} R^r{}_i - \frac{1}{n-1} R(A_j R_{ki} - A_r R^r{}_i g_{jk}) - A_r R^r{}_s R^s{}_i g_{jk}$$
$$+ A_i R_{rk} R^r{}_j - \frac{1}{n-1} R(A_i R_{kj} - A_r R^r{}_j g_{ik}) - A_r R^r{}_s R^s{}_j g_{ik} = 0.$$

Contracting (40) with g^{jk}, we find

$$A_r R^r{}_s R^s{}_i = \frac{1}{n-1} R A_r R^r{}_i + \frac{1}{n} R^{rs} R_{rs} A_i - \frac{1}{n(n-1)} R^2 A_i,$$

which reduces (40) to the fotm

$$\left(R_{rk} R^r{}_i - \frac{1}{n-1} R R_{ik} - \frac{1}{n} R^{rs} R_{rs} g_{ik} + \frac{1}{n(n-1)} R^2 g_{ik}\right) A_j$$
$$+\left(R_{rk} R^r{}_j - \frac{1}{n-1} R R_{jk} - \frac{1}{n} R^{rs} R_{rs} g_{jk} + \frac{1}{n(n-1)} R^2 g_{jk}\right) A_i = 0.$$

Since A_j does not identically vanish and M is analytic, the last result leads immediately to (36). This completes the proof.

THEOREM 4. *Let M be an analytic n-dimensional ($n \geqslant 3$) semi-Ricci--symmetric manifold of harmonic curvature. If M admits a conformal collineation which is not affine, then the Ricci tensor of M is parallel or it satisfies the condition*

$$(41)\qquad R^{rs} R_{rs} = \frac{3n-4}{4(n-1)^2} R^2.$$

Proof. Condition (37) implies $R = \text{const}$. Differentiating (36) covariantly and making use of (37), we get

$$(42)\qquad R_{ri,k} R^r{}_j + R_{ri} R^r{}_{k,j} = \frac{1}{n-1} R R_{ij,k} + \frac{2}{n} R^{rs} R_{rs,k} g_{ij},$$

whence, by contraction with g^{ik}, we obtain $R^{rs} R_{rs,j} = 0$.

The last result reduces (42) to the form

$$(43)\qquad R_{ri,k} R^r{}_j + R_{ri} R^r{}_{k,j} = \frac{1}{n-1} R R_{ij,k}.$$

But (43), together with (37), implies $R_{ri,k} R^r{}_j = R_{ri,j} R^r{}_k$, whence

$$R_{ri,k} R^r{}_j = R_{ri,j} R^r{}_k = R_{rj,i} R^r{}_k = R_{rj,k} R^r{}_i = R_{ri} R^r{}_{j,k}.$$

Thus, by (43) and (37), we have

$$R_{ri}R^{r}{}_{j,k} = \frac{1}{2(n-1)}RR_{ij,k} \tag{44}$$

and, consequently,

$$R_{sp}R^{s}{}_{r}R^{r}{}_{j,k} = \frac{1}{2(n-1)}RR_{rp}R^{r}{}_{j,k}. \tag{45}$$

Applying now (36) and (44) to (45), we obtain easily

$$\left(R^{rs}R_{rs} - \frac{3n-4}{4(n-1)^2}R^2\right)R_{pj,k} = 0.$$

This completes the proof.

THEOREM 5. *Let M be an n-dimensional ($n \geqslant 3$) analytic manifold of harmonic curvature with a positive definite metric. If M satisfies* (7) *and admits a conformal collineation which is not affine, then the Ricci tensor of M is parallel.*

Proof. Assume (41) holds. Then

$$\left(R^{ij} - \frac{1}{2(n-1)}Rg^{ij}\right)\left(R_{ij} - \frac{1}{2(n-1)}Rg_{ij}\right) = 0.$$

But the last result implies

$$R_{ij} = \frac{1}{2(n-1)}Rg_{ij}.$$

Hence $R_{ij} = 0$. This completes the proof.

Remark 5. Lemma 4 of [14] states that the Ricci tensor of a compact, with positive definite metric, semi-Ricci-symmetric manifold of harmonic curvature is necessarily parallel.

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